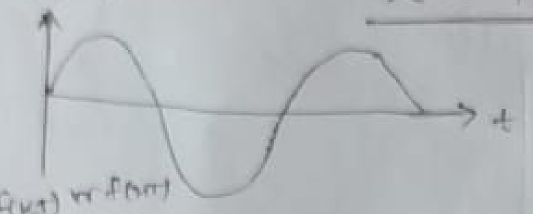
Z - Transform

$f(t) = A \sin \omega t$



LT & Fourier Transform both are continous functions

$f(t)$ or $f(kT)$ or $f(nT)$

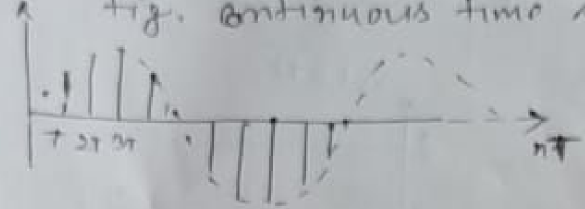


fig. continuous time signal

Z→Transform for discrete function (digital form)

$t = nT$ { where n or k discrete number
$t = kT$ { where T → Samping Period.

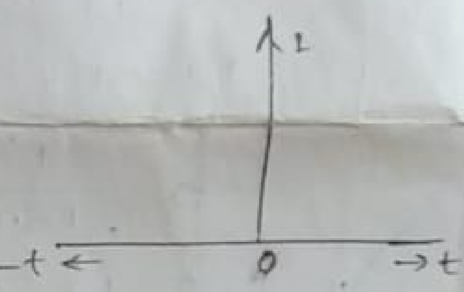
$Z = f(kT)$ Z→Transform of a discrete time signal $f(kT)$ is defined as.

$$\boxed{Z[f(kT)] = \sum_{k=0}^{\infty} f(kT) z^{-k}}$$

or

$$Z[f(kT)] = \sum_{k=-\infty}^{\infty} f(kT) z^{-k}$$

① Unit impulse function:



$\delta(t) = 1$; at $t = 0$

$= 0$; otherwise for $t > 0, t < 0$.

Put $t = kT$

$\delta(kT) = 1$ for $k = 0$

$= 0$ for $k < 0$

$$Z[\delta(kT)] = \sum_{k=0}^{\infty} \delta(kT) z^{-k}$$

$$= \delta(0) z^{-0} + \sum_{k=1}^{\infty} \delta(kT) z^{-k}$$

due to impulse

$$= \delta(0) z^{-0} + \delta(T) z^{-1} + \delta(2T) z^{-2}$$

$$= 1 \cdot z^{-0} + \text{(higher order \& Lower order not inclu}$$

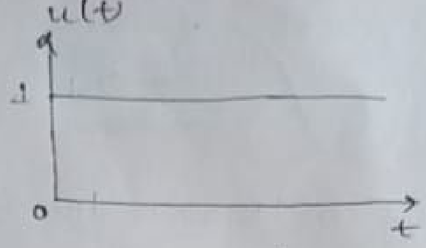
$$Z[\delta(kT)] = 1.$$

1

2) Unit step function:

$u(t) = 1 \quad t \geq 0$

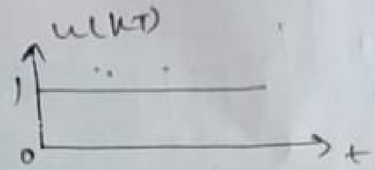
$\quad = 0 \quad t < 0$



put $t = kT$

$\rightarrow u(kT) = 1 \quad$ for $k \geq 0$

$\quad = 0 \quad$ otherwise $k < 0$.



$$Z[u(kT)] = \sum_{k=0}^{\infty} u(kT) z^{-k}$$

$u(t) = 1$
$u(kT) = 1$
$u(kT) z^{-k}$
$u(0) z^{-0}$
$1 \cdot z^{-0} = 1.$

$$= 1 \cdot z^{-0} + 1 \cdot z^{-1} + 1 \cdot z^{-2} + \cdots$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

we know that discrete time unit step function $u(kT)$ exists only for +ve value of $k(t)$ i.e it is causal.

$1 + x + x^2 + \cdots$ G.P

$\Rightarrow \dfrac{1}{1 - c.r} \rightarrow c.r \rightarrow$ common ration

$a = 1, \quad c.r = z^{-1}.$

$$Z[u(kT)] = \frac{1}{1 - z^{-1}} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z - 1}$$

$\because \left|\frac{1}{z}\right| < 1$

$\because \frac{1}{z} < 1$

3) $f(kT) = a^{kT}$

$f(k) = a^{k}$

$$Z[f(kT)] = Z[f(k)] = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} a^{k} \cdot z^{-k}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots$$

$a = 1, \quad c.r = az^{-1}$

$$Z[a^k] = \frac{1}{1 - az^{-1}} = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}$$

$\left|\frac{a}{z}\right| < 1$

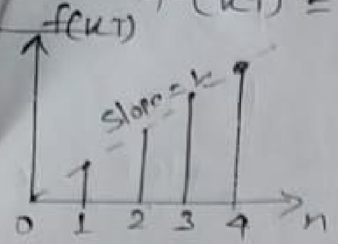
$|a| < z$

$$\boxed{Z[a^k] = \frac{z}{z - a}}$$

2

(4) Unit ramp sequence (Ramp function)

$$f(kT) = r(kT) = kT\, u(kT)$$



$$F(z) = \sum_{k=0}^{\infty} (kT) z^{-k}$$

$$= 0 + Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots$$

$$= Tz^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots)$$

$$= Tz^{-1}\big(\underbrace{1 + z^{-1} + z^{-2} + z^{-3} + \cdots}$$
$$\underbrace{+z^{-1} + z^{-2} + z^{-3} + \cdots}$$
$$\underbrace{+z^{-2} + z^{-3} + \cdots}$$
$$\underbrace{+z^{-3} + \cdots}$$

$$= Tz^{-1}\left(\frac{1}{1-z^{-1}} + \frac{z^{-1}}{1-z^{-1}} + \frac{z^{-2}}{1-z^{-1}} + \cdots\right)$$

$$= Tz^{-1}\left(\frac{1+z^{-1}+z^{-2}}{1-z^{-1}}\right)$$

$$= \frac{Tz^{-1}}{(1-z^{-1})}(1 + z^{-1} + z^{2} + \cdots) \qquad \text{h.p} = \frac{a}{1-\text{c.r}} = \frac{a}{a-\text{c}r}$$

$$= \frac{1}{1-z^{-1}} = \frac{1}{1-\frac{1}{z}}$$

$$= \frac{Tz^{-1}}{1-z^{-1}} \times \frac{1}{(1-z^{-1})}$$

$$= \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{\frac{T}{z}}{\left(1-\frac{1}{z}\right)^2} = \frac{\frac{T}{z}}{\frac{(z-1)^2}{z^2}} = \frac{Tz}{(z-1)^2}$$

$$\boxed{z[f(kT)] = \frac{Tz}{(z-1)^2}}$$

(5) $f(t) = e^{j\omega t} \cdot u(t) \rightarrow$ continuous function

$f(kT) = e^{j\omega kT} \cdot u(kT) \rightarrow$ discrete function.

$$z[f(kT)] = z[e^{j\omega kT} \cdot u(kT)]$$

$$= \sum_{k=0}^{\infty} e^{j\omega kT} \cdot u(kT) \cdot z^{-k}$$

$$= 1 \cdot z^{-0} + e^{j\omega T} \cdot z^{-1} + e^{j\omega 2T} \cdot z^{-2} + \cdots$$

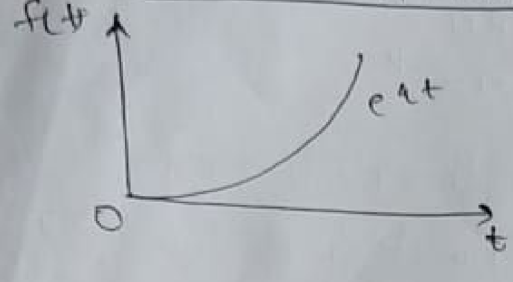
$a = 1$, $\text{c.r} = e^{j\omega T} \cdot z^{-1}$

$$z[f(kT)] = \frac{1}{1-e^{j\omega T} \cdot z^{-1}} = \frac{1}{1-\frac{e^{j\omega T}}{z}} = \frac{z}{z-e^{j\omega T}}$$

$$z[f(kT)] = z[e^{j\omega kT} \cdot u(kT)] = \frac{z}{z-e^{j\omega T}}.$$

3

(6) Parabolic function:



$$f(t) = e^{at} \quad t \geq 0$$
$$= 0 \quad t < 0.$$

$$Z[e^{akT}] = \sum_{k=0}^{\infty} e^{akT} \cdot z^{-k}$$
$$= 1 + e^{aT} z^{-1} + e^{2aT} z^{-2} + \cdots$$
$$= \frac{1}{1 - e^{aT} \cdot z^{-1}} = \frac{z}{z - e^{aT}}$$

Similarly, for an exponentially decaying function, e^{-at} the z-Transform will be

$$Z[e^{-at}] = \sum_{k=0}^{\infty} e^{-akT} z^{-k}$$
$$= \frac{z}{z - e^{-aT}}$$

As special case:

Take $a = 1$

$$z[e^{-t}] = \sum_{k=0}^{\infty} e^{-kT} \cdot z^{-k}$$
$$= \frac{z}{z - e^{-T}}$$

Assuming Sampling time $T = 1$ sec. we get

$$z[e^{-t}] = \frac{z}{z - e^{-1}}$$
$$= 1 + 0.9512 z^{-1} + 0.9048 z^{-2} + \cdots$$

(5.)

$$f(t) = \sin n\omega t$$

$$f(kT) = \sin \omega kT \quad ; \quad k \geq 0$$

$$= 0 \quad ; \quad k < 0.$$

$$z[\sin \omega kT] = \sum_{k=0}^{\infty} \sin \omega kT \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{j\omega kT} - e^{-j\omega kT}}{2j} \right) z^{-k}$$

$$= \frac{1}{2j} \left[\sum_{k=0}^{\infty} e^{j\omega kT} z^{-k} - \sum_{k=0}^{\infty} e^{-j\omega kT} \cdot z^{-k} \right]$$

$$= \frac{1}{2j} \left[\frac{z}{(z - e^{j\omega T})} - \frac{z}{(z - e^{-j\omega T})} \right]$$

$$= \frac{z}{2j} \left[\frac{\cancel{z} - e^{-j\omega T} - \cancel{z} + e^{j\omega T}}{(z - e^{j\omega T})(z - e^{-j\omega T})} \right]$$

$$= \frac{z}{2j} \left[\frac{e^{j\omega T} - e^{-j\omega T}}{(z - e^{j\omega T})(z - e^{-j\omega T})} \right]$$

$$= \frac{z}{2j} \left(\frac{e^{j\omega T} - e^{-j\omega T}}{z^2 - 2z \cos \omega T + 1} \right)$$

$$= \frac{z}{z^2 - 2z \cos \omega T + 1} \left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j} \right]$$

$$z[\sin \omega kT] = \frac{z \times \sin \omega T}{z^2 - 2z \cos \omega T + 1} = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$$

$$z - e^{j\omega T}$$
$$z - e^{-j\omega T}$$

$$z^2 - z e^{j\omega T} - z e^{-j\omega T} + e^{j\omega T} \cdot e^{-j\omega T}$$

$$z^2 - z(e^{j\omega T} + e^{-j\omega T}) + e^{j\omega T - j\omega T}$$

$$z^2 - z(e^{j\omega T} + e^{-j\omega T}) + e^0$$

$$z^2 - 2z\left(\frac{e^{j\omega T} + e^{-j\omega T}}{2}\right) + 1$$

$$z^2 - 2z \cos \omega T + 1$$

$$\sin \omega kT = \frac{e^{j\omega kT} - e^{-j\omega kT}}{2j}$$

$$\cos \omega kT = \frac{e^{j\omega kT} + e^{-j\omega kT}}{2}$$

5

(6) Given that $f(t) = e^{-at}\sin\omega t$

The discrete sequence $f(k)$ is generated by replacing t by kT, where T is the sampling time period.

$$\therefore f(k) = e^{-akT}\sin\omega kT$$

By the definition of one sided z-Transform we get

$$F(z) = Z\{f(k)\}$$

$$= \sum_{k=0}^{\infty} e^{-akT}\sin\omega kT \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{-akT}\left(\frac{e^{j\omega kT} - e^{-j\omega kT}}{2j}\right) z^{-k}$$

$$= \frac{1}{2j}\sum_{k=0}^{\infty}\left(e^{-aT}\cdot e^{j\omega T} z^{-1}\right)^k - \frac{1}{2j}\sum_{k=0}^{\infty}\left(e^{-aT}\cdot e^{-j\omega T} z^{-1}\right)^k$$

From infinite geometric sum series formula we know that $\sum_{k=0}^{\infty} c^k = \frac{1}{1-c \cdot r}$

$$F(z) = \frac{1}{2j}\cdot\frac{1}{1-e^{-aT}\cdot e^{j\omega T} z^{-1}} - \frac{1}{2j}\cdot\frac{1}{1-e^{-aT}\cdot e^{-j\omega T} z^{-1}}$$

$$= \frac{1}{2j}\cdot\frac{1}{1-\frac{e^{j\omega T}}{ze^{aT}}} - \frac{1}{2j}\cdot\frac{1}{1-\frac{e^{-j\omega T}}{ze^{aT}}}$$

$$= \frac{1}{2j}\cdot\frac{ze^{aT}}{ze^{aT}-e^{j\omega T}} - \frac{1}{2j}\cdot\frac{ze^{aT}}{ze^{aT}-e^{-j\omega T}}$$

$$= \frac{1}{2j}\left[\frac{ze^{aT}(ze^{aT}-e^{-j\omega T}) - ze^{aT}(ze^{aT}-e^{j\omega T})}{(ze^{aT}-e^{j\omega T})(ze^{aT}-e^{-j\omega T})}\right]$$

$$= \frac{1}{2j}\left[\frac{ze^{aT}\left[\cancel{ze^{aT}} - e^{-j\omega T} - \cancel{ze^{aT}} + e^{j\omega T}\right]}{(ze^{aT})^2 - ze^{aT}\cdot e^{-j\omega T} - ze^{aT}\cdot e^{j\omega T} + e^{j\omega T}\cdot e^{-j\omega T}}\right]$$

$$= \frac{ze^{aT}\left(\frac{e^{j\omega T}-e^{-j\omega T}}{2j}\right)}{z^2e^{2aT} - ze^{aT}\left(e^{j\omega T}+e^{-j\omega T}\right)+1}$$

$$Z\left[e^{-akT}\sin\omega kT\right] = \frac{ze^{aT}\cdot\sin\omega T}{z^2e^{2aT} - 2ze^{aT}\cos\omega T + 1}$$

$$\cos\omega T = \frac{e^{j\omega T}+e^{-j\omega T}}{2}$$

6