Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson 13

The Three-Moment Equations-Ii

Instructional Objectives

After reading this chapter the student will be able to
1. Derive three-moment equations for a continuous beam with yielding supports.
2. Write compatibility equations of a continuous beam in terms of three moments.
3. Compute reactions in statically indeterminate beams using three-moment equations.
4. Analyse continuous beams having different moments of inertia in different spans and undergoing support settlements using three-moment equations.

13.1 Introduction

In the last lesson, three-moment equations were developed for continuous beams with unyielding supports. As discussed earlier, the support may settle by unequal amount during the lifetime of the structure. Such future unequal settlement induces extra stresses in statically indeterminate beams. Hence, one needs to consider these settlements in the analysis. The three-moment equations developed in the pervious lesson could be easily extended to account for the support yielding. In the next section three-moment equations are derived considering the support settlements. In the end, few problems are solved to illustrate the method.

13.2 Derivation of Three-Moment Equation

Consider a two span of a continuous beam loaded as shown in Fig.13.1. Let M_L, M_C and M_R be the support moments at left, center and right supports respectively. As stated in the previous lesson, the moments are taken to be positive when they cause tension at the bottom fibers. I_L and I_R denote moment of inertia of left and right span respectively and l_L and l_R denote left and right spans respectively. Let δ_L, δ_C and δ_R be the support settlements of left, centre and right supports respectively. δ_L, δ_C and δ_R are taken as negative if the settlement is downwards. The tangent to the elastic curve at support C makes an angle θ_{CL} at left support and θ_{CR} at the right support as shown in Fig. 13.1. From the figure it is observed that,

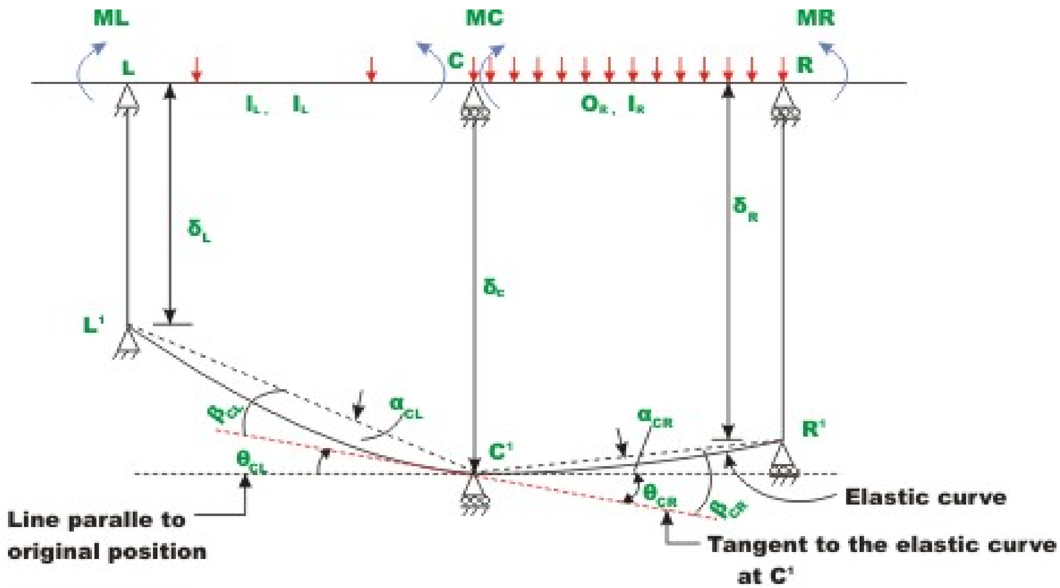


Fig. 13.1 Continuous beam with support settlement

$$\theta_{CL} = \theta_{CR} \quad (13.1)$$

The rotations β_{CL} and β_{CR} due to external loads and support moments are calculated from the M/EI diagram .They are (see lesson 12)

$$\beta_{CL} = \frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_L l_L}{6EI_L} + \frac{M_C l_L}{3EI_L} \quad (13.2a)$$

$$\beta_{CR} = \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_R l_R}{6EI_R} + \frac{M_C l_R}{3EI_R} \quad (13.2b)$$

The rotations of the chord $L'C'$ and $C'R'$ from the original position is given by

$$\alpha_{CL} = \frac{\delta_L - \delta_C}{l_L} \quad (13.3a)$$

$$\alpha_{CR} = \frac{\delta_R - \delta_C}{l_R} \quad (13.3b)$$

From Fig. 13.1, one could write,

$$\theta_{CL} = \alpha_{CL} - \beta_{CL} \tag{13.4a}$$

$$\theta_{CR} = \beta_{CR} - \alpha_{CR} \tag{13.4b}$$

Thus, from equations (13.1) and (13.4), one could write,

$$\alpha_{CL} - \beta_{CL} = \beta_{CR} - \alpha_{CR} \tag{13.5}$$

Substituting the values of $\alpha_{CL}, \alpha_{CR}, \beta_{CL}$ and β_{CR} in the above equation,

$$M_L\left(\frac{l_L}{I_L}\right) + 2M_C\left\{\frac{l_L}{I_L} + \frac{l_R}{I_R}\right\} + M_R\left(\frac{l_R}{I_R}\right) = -\frac{6A_R\bar{x}_R}{I_R l_R} - \frac{6A_L\bar{x}_L}{I_L l_L} + 6E\left(\frac{\delta_L - \delta_C}{l_L}\right) + 6E\left(\frac{\delta_R - \delta_C}{l_R}\right)$$

This may be written as

$$M_L\left(\frac{l_L}{I_L}\right) + 2M_C\left\{\frac{l_L}{I_L} + \frac{l_R}{I_R}\right\} + M_R\left(\frac{l_R}{I_R}\right) = -\frac{6A_R\bar{x}_R}{I_R l_R} - \frac{6A_L\bar{x}_L}{I_L l_L} - 6E\left[\left(\frac{\delta_C - \delta_L}{l_L}\right) + \left(\frac{\delta_C - \delta_R}{l_R}\right)\right]$$
(13.6)

The above equation relates the redundant support moments at three successive spans with the applied loading on the adjacent spans and the support settlements.

Example 13.1

Draw the bending moment diagram of a continuous beam BC shown in Fig.13.2a by three moment equations. The support B settles by 5mm below A and C. Also evaluate reactions at A, B and C. Assume EI to be constant for all members and $E = 200$ GPa, $I = 8 \times 10^6$ mm^4

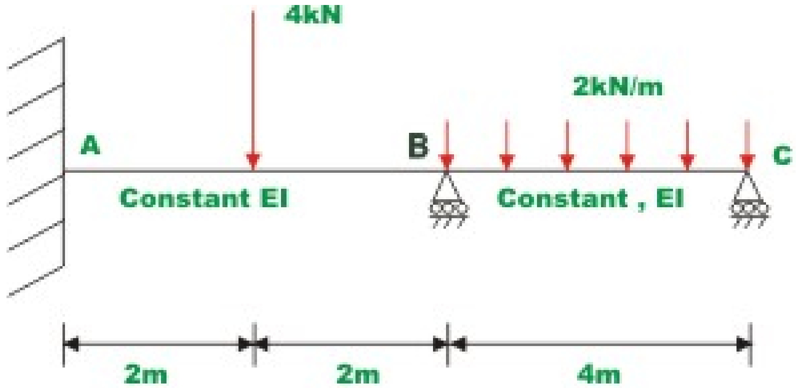


Fig. 13.2(a) Example 13.1

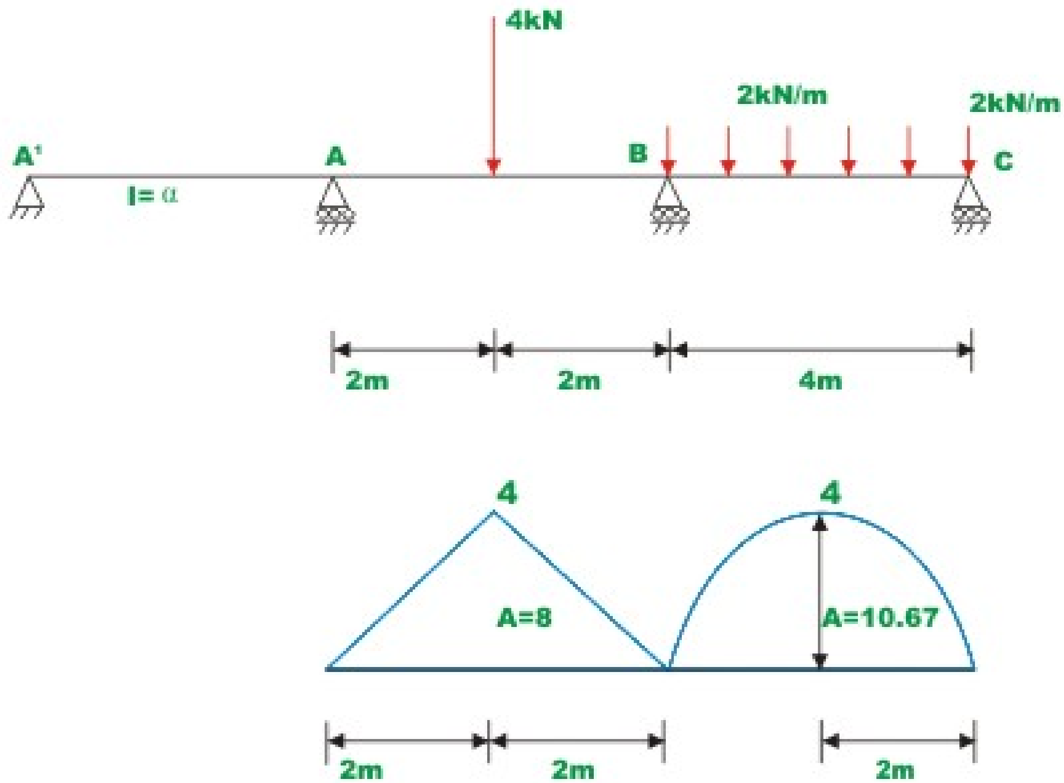


Fig. 13.2(b) Bending moment diagram due to applied loading

Assume an imaginary span having infinitely large moment of inertia and arbitrary span L' left of A as shown in Fig.13.2b .Also it is observed that moment at C is zero.

The given problem is statically indeterminate to the second degree. The moments M_A and M_B ,the redundants need to be evaluated. Applying three moment equation to the span *A'AB*,

$$\delta_L = \delta_C = 0 \text{ and } \delta_R = -5\times10^{-3}\, m$$

$$M'_A\left(\frac{L'}{\infty}\right) + 2M_A\left\{\frac{L'}{\infty} + \frac{4}{I}\right\} + M_B\left(\frac{4}{I}\right) = -\frac{6\times8\times2}{I(4)} - 6E\left(0 + \frac{0-(-5\times10^{-3}}{4}\right)$$

$$8M_A + 4M_B = -24 - 6EI\times\frac{5\times10^{-3}}{4} \qquad (1)$$

Note that, $EI = 200\times10^9 \times \frac{8\times10^6 \times10^{-12}}{10^3} = 1.6\times10^3\,\text{kNm}^2$

Thus,

$$8M_A + 4M_B = -24 - 6\times1.6\times10^3\times\frac{5\times10^{-3}}{4}$$

$$8M_A + 4M_B = -36 \qquad (2)$$

Again applying three moment equation to span ABC the other equations is obtained. For this case, $\delta_L = 0$, $\delta_C = -5\times10^{-3}\, m$ (negative as the settlement is downwards) and $\delta_R = 0$.

$$M_A\left\{\frac{4}{I}\right\} + 2M_B\left\{\frac{4}{I} + \frac{4}{I}\right\} = -\frac{24}{I} - \frac{6\times10.667\times2}{I\times4} - 6E\left(\frac{-5\times10^{-3}}{4} - \frac{5\times10^{-3}}{4}\right)$$

$$4M_A + 16M_B = -24 - 32 + 6\times1.6\times10^3\times\frac{10\times10^3}{4}$$

$$4M_A + 16M_B = -32 \qquad (3)$$

Solving equations (2) and (3),

$$M_B = -1.0 \text{ kN.m}$$
$$M_A = -4.0 \text{ kN.m} \qquad (4)$$

Now, reactions are calculated from equations of static equilibrium (see Fig.13.2c).

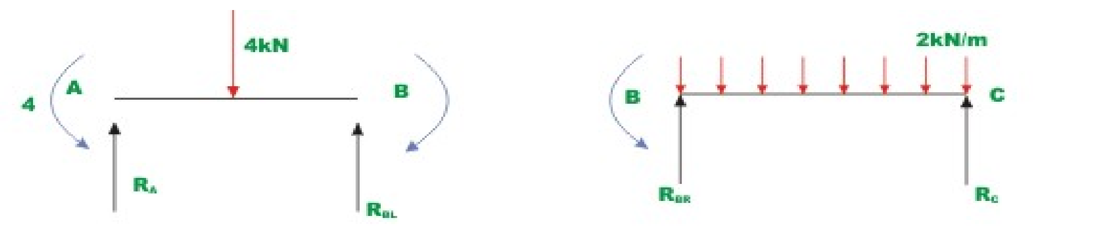
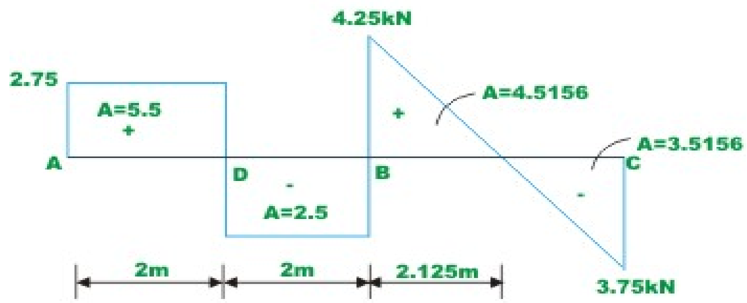
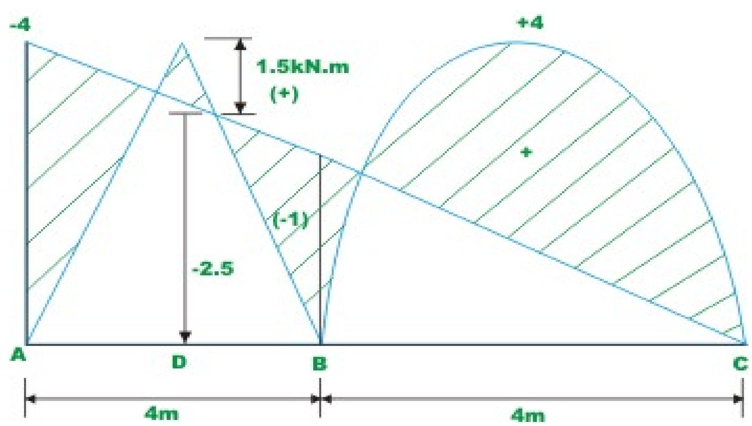


Fig.13.2 (c) Free - body diagram of two members



Shear force diagram



Bending moment diagram

Fig.13.2(d) Shear force and bending moment diagram

Thus,

$$R_A = 2.75\,\text{kN}\ (\uparrow)$$
$$R_{BL} = 1.25\,\text{kN}\ (\uparrow)$$
$$R_{BR} = 4.25\,\text{kN}\ (\uparrow)$$
$$R_C = 3.75\,\text{kN}\ (\uparrow)$$

The reactions at B,

$$R_B = R_{BR} + R_{BL} = 5.5\,\text{kN} \tag{5}$$

The area of each segment of the shear force diagram for the given continuous beam is also indicated in the above diagram. This could be used to verify the previously computed moments. For example, the area of the shear force diagram between A and B is $5.5\ \text{kN.m}$.This must be equal to the change in the bending moment between A and D, which is indeed the case ($-4-1.5=5.5\ \text{kN.m}$). Thus, moments previously calculated are correct.

Example 13.2

A continuous beam $ABCD$ is supported on springs at supports B and C as shown in Fig.13.3a. The loading is also shown in the figure. The stiffness of springs is $k_B = \frac{EI}{20}$ and $k_C = \frac{EI}{30}$ .Evaluate support reactions and draw bending moment diagram. Assume EI to be constant.

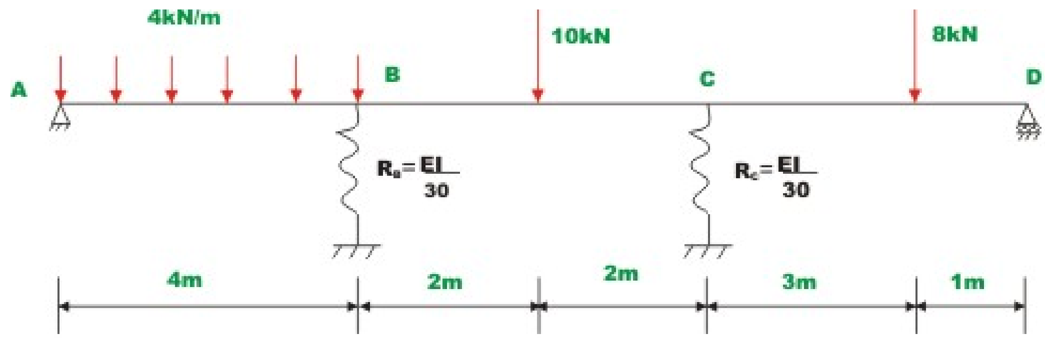


Fig.13.3(a) Continuous beam of Example 13.2

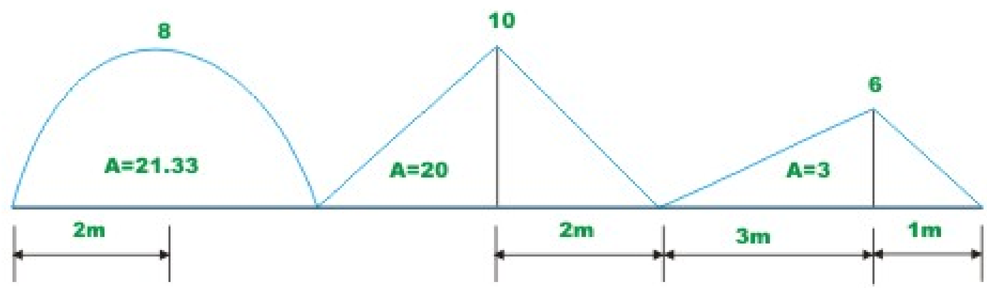


Fig.13.3(b) Bending moment diagram on simple spans due to applied loading

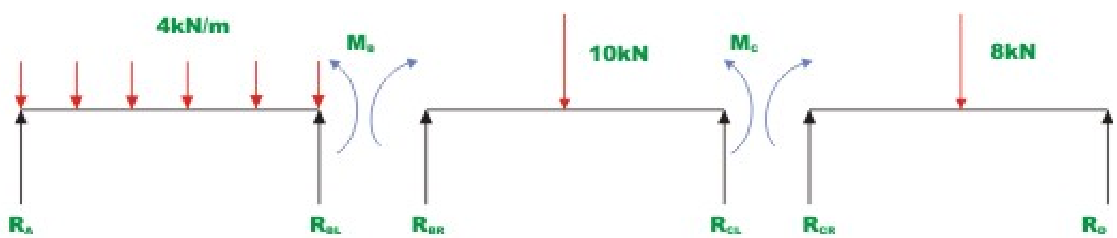


Fig.13.3(c) Computation of reactions

In the given problem it is required to evaluate bending moments at supports B and C. By inspection it is observed that the support moments at A and D are zero. Since the continuous beam is supported on springs at B and C, the support settles. Let R_B and R_C be the reactions at B and C respectively. Then the support settlement at B and C are $\frac{R_B}{k_B}$ and $\frac{R_C}{k_C}$ respectively. Both the settlements are negative and in other words they move downwards. Thus,

$$\delta_A = 0, \delta_B = \frac{-20R_B}{EI}, \delta_C = \frac{-30R_C}{EI} \text{ and } \delta_D = 0 \qquad (1)$$

Now applying three moment equations to span ABC (see Fig.13.2b)

$$M_A\left\{\frac{4}{I}\right\}+2M_B\left\{\frac{4}{I}+\frac{4}{I}\right\}+M_C\left\{\frac{4}{I}\right\}=-\frac{6\times 21.33\times 2}{I\times 4}-\frac{6\times 20\times 2}{I\times 4}-6E\left[\frac{-20R_B}{4EI}+\frac{\frac{-20R_B}{EI}+\frac{30R_C}{EI}}{4}\right]$$

Simplifying,

$$16M_B+4M_C=-124+60R_B-45R_C \qquad (2)$$

Again applying three moment equation to adjacent spans BC and CD,

$$M_B\left\{\frac{4}{I}\right\}+2M_C\left\{\frac{4}{I}+\frac{4}{I}\right\}=-\frac{60}{I}-\frac{(6\times 9\times 2+6\times 3\times \frac{2}{3}\times 1)}{I\times 4}-6E\left[\frac{-\frac{30R_C}{EI}+\frac{20R_B}{EI}}{4}+\frac{-30R_C}{4EI}\right]$$

$$4M_B+16M_C=-90+90R_C-30R_B \qquad (3)$$

In equation (2) and (3) express R_B and R_C in terms of M_B and M_C (see Fig.13.2c)

$$\begin{aligned}
R_A &= 8+0.25M_B \quad (\uparrow)\\
R_{BL} &= 8-0.25M_B \quad (\uparrow)\\
R_{BR} &= 5+0.25M_C-0.25M_B \quad (\uparrow)\\
R_{CL} &= 5+0.25M_B-0.25M_C \quad (\uparrow)\\
R_{CR} &= 2-0.25M_C \quad (\uparrow)\\
R_D &= 6+0.25M_C \quad (\uparrow)
\end{aligned} \qquad (4)$$

Note that initially all reactions are assumed to act in the positive direction (i.e. upwards) .Now,

$$R_B=R_{BL}+R_{BR}=13-0.5M_B+0.25M_C$$

$$R_C=R_{CL}+R_{CR}=7+0.25M_B-0.5M_C \qquad (5)$$

Now substituting the values of R_B and R_C in equations (2) and (3),

$$16M_B+4M_C=-124+60(13-0.5M_B+0.25M_C)-45(7+0.25M_B-0.5M_C)$$

Or,

$$57.25M_B - 33.5M_C = 341 \qquad (6)$$

And from equation 3,

$$4M_B + 16M_C = -90 + 90(7 + 0.25M_B - 0.5M_C) - 30(13 - 0.5M_B + 0.25M_C)$$

Simplifying,

$$-33.5M_B + 68.5M_C = 150 \qquad (7)$$

Solving equations (6) and (7)

$$\begin{aligned} M_C &= 7.147 \text{ kN.m} \\ M_B &= 10.138 \text{ kN.m} \end{aligned} \qquad (8)$$

Substituting the values of M_B and M_C in (4),reactions are obtained.

$$\begin{array}{ll} R_A = 10.535 \text{ kN } (\uparrow) & R_{BL} = 5.465 \text{ kN } (\uparrow) \\ R_{BR} = 4.252 \text{ kN } (\uparrow) & R_{CL} = 5.748 \text{ kN } (\uparrow) \\ R_{CR} = 0.213 \text{ kN } (\uparrow) & R_D = 7.787 \text{ kN } (\uparrow) \\ R_B = 9.717 \text{ kN } (\uparrow) \; \textit{and} & R_C = 5.961 \text{ kN } (\uparrow) \end{array}$$

The shear force and bending moment diagram are shown in Fig. 13.2d.

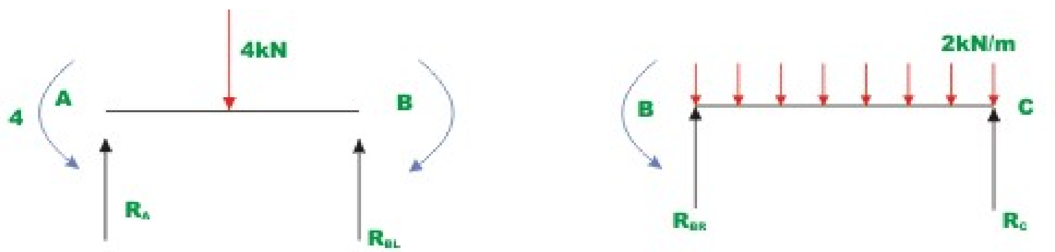
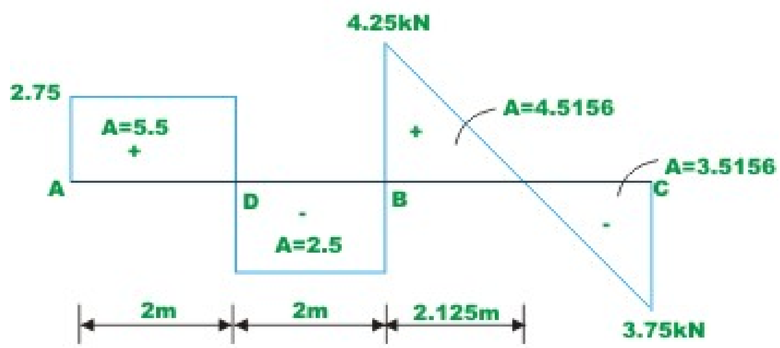
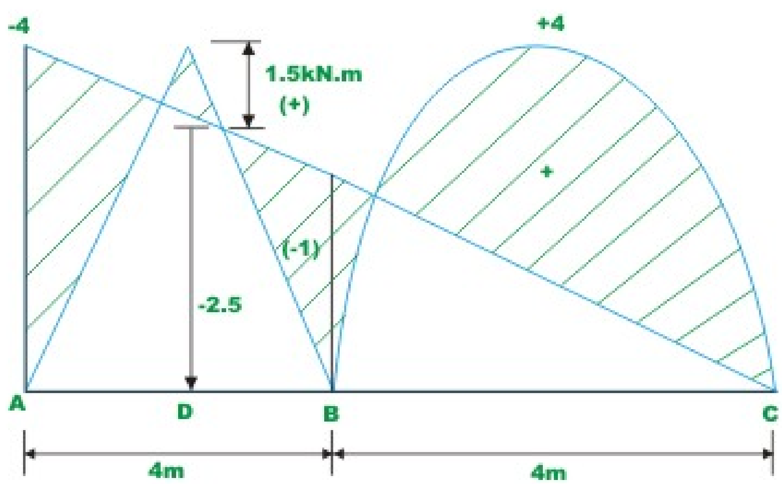


Fig.13.2 (c) Free - body diagram of two members



Shear force diagram



Bending moment diagram

Fig.13.2(d) Shear force and bending moment diagram

Example 13.3

Sketch the deflected shape of the continuous beam *ABC* of example 13.1.
The redundant moments M_A and M_B for this problem have already been computed in problem 13.1.They are,

$$M_B = -1.0 \text{ kN.m}$$
$$M_A = -4.0 \text{ kN.m}$$

The computed reactions are also shown in Fig.13.2c.Now to sketch the deformed shape of the beam it is required to compute rotations at *B* and *C*. These joints rotations are computed from equations (13.2) and (13.3).
For calculating θ_A, consider span *A'AB*

$$\theta_A = \beta_{AR} - \alpha_{AR}$$

$$= \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_B l_R}{6EI_R} + \frac{M_A l_R}{3EI_R} - \left(\frac{\delta_B - \delta_A}{4}\right)$$

$$= \frac{6 \times 8 \times 2}{1.6 \times 10^3 \times 4} + \frac{M_B \times 4}{1.6 \times 10^3 \times 6} + \frac{M_A \times 4}{1.6 \times 10^3 \times 3} - \left(\frac{\delta_B - \delta_A}{4}\right)$$

$$= \frac{6 \times 8 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 6} + \frac{(-4) \times 4}{1.6 \times 10^3 \times 3} + \left(\frac{5 \times 10^{-3}}{4}\right)$$

$$= 0 \qquad (1)$$

For calculating θ_{BL}, consider span *ABC*

$$\theta_{BL} = \alpha_{BL} - \beta_{BL}$$

$$= -\left(\frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_A l_L}{6EI_L} + \frac{M_B l_L}{3EI_L}\right) + \left(\frac{\delta_A - \delta_B}{l_L}\right)$$

$$= -\left(\frac{8 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-4) \times 4}{1.6 \times 10^3 \times 6} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 3}\right) + \left(\frac{5 \times 10^3}{4}\right)$$

$$= 1.25 \times 10^{-3} \text{ radians} \qquad (2)$$

For θ_{BR} consider span ABC

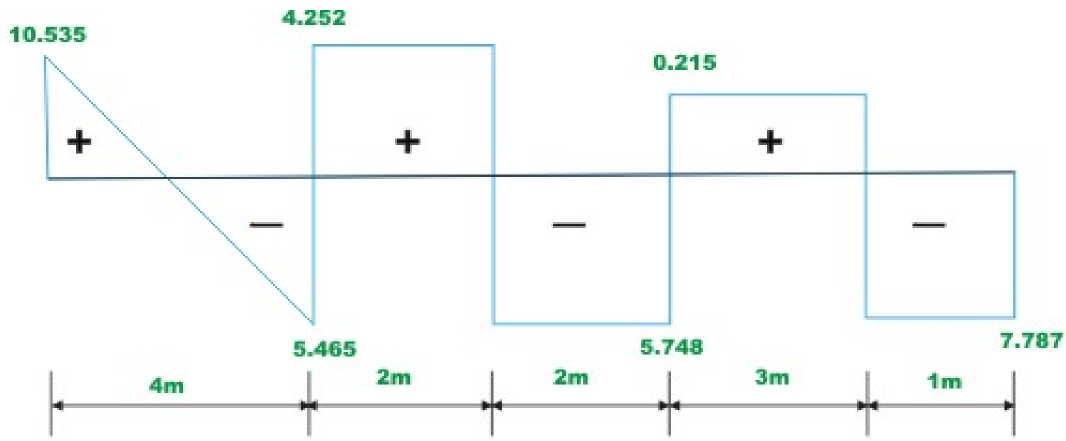
$$\theta_{BR} = \left(\frac{10.67 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 3} \right) - \left(0 + \frac{5 \times 10^3}{4} \right)$$

$$= -1.25 \times 10^{-3} \text{ radians} \qquad (3)$$

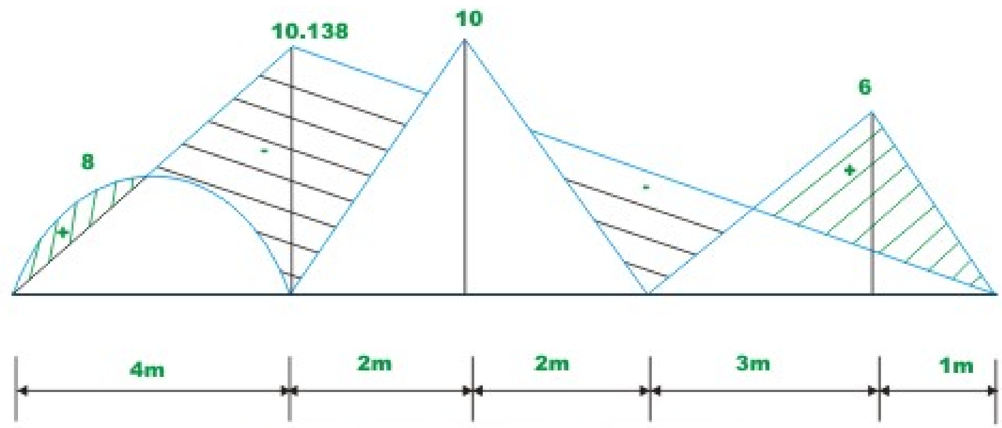
$$\theta_C = -\left(\frac{10.67 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 3} \right) - \left(\frac{\delta_B - \delta_C}{4} \right)$$

$$= -3.75 \times 10^{-3} \text{ radians.} \qquad (4)$$

The deflected shape of the beam is shown in Fig. 13.4.



Shear force diagram



Bending moment diagram
Fig.13.3(d)

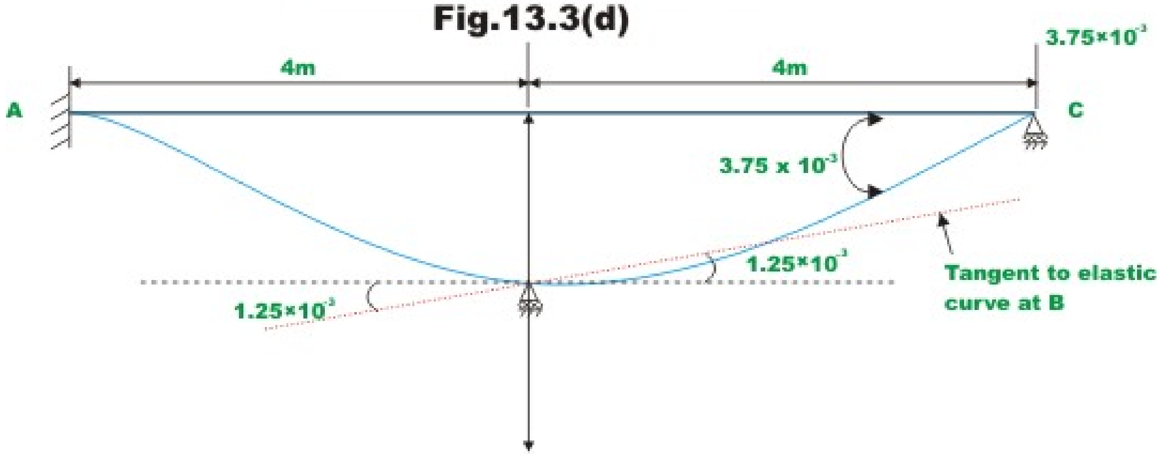


Fig.13.4(a) Elastic curve Example 13.3

Summary

The continuous beams with unyielding supports are analysed using three-moment equations in the last lesson. In this lesson, the three-moment-equations developed in the previous lesson are extended to account for the support settlements. The three-moment equations are derived for the case of a continuous beam having different moment of inertia in different spans. Few examples are derived to illustrate the procedure of analysing continuous beams undergoing support settlements using three-moment equations.