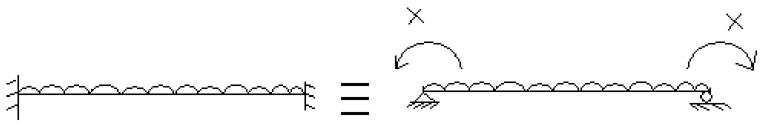
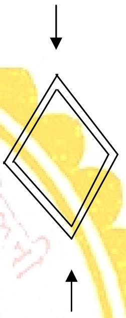
7. MATRIX FORCE METHOD

7.1 INTRODUCTION

Conventional force method.





Structure as a whole or any substructure

↓

Must Satisfy

↓

1. Equilibrium of forces.
2. Displacement compatibility.
3. Force-displacement relation.

Matrix Force Method – also called as Flexibility method.
Member forces are treated as the basic unknowns.
Similar to the classical force method, but based on matrix approach.
Based on **<u>finite element</u>** concept

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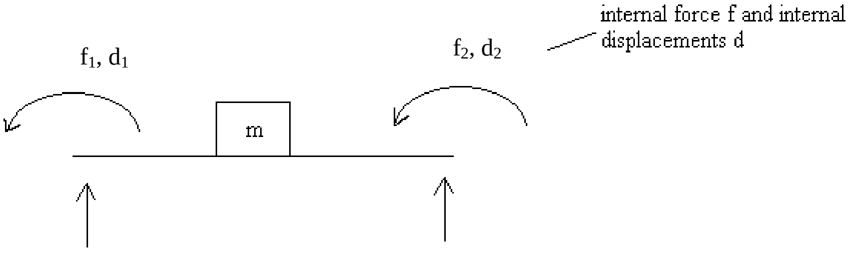
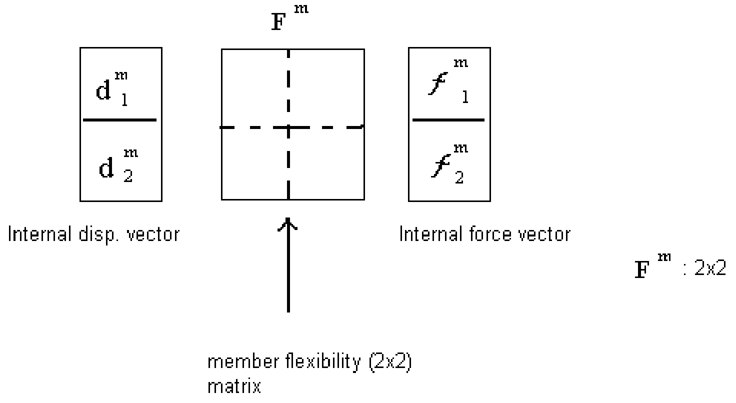
Step-by-step building up of force-displacement relationship using basic elements composing the structure.

7.2 ASSUMPTIONS

- Hooke's law.
- Small deflections.
- Change in length under a deflection ⊥ to member length = 0.
- Principal of superposition.
- Frames – member inextensible.

7.3 FLEXIBILITY MATRIX FOR A FRAME / BEAM ELEMENT





Shears not included since dependent on moments.

F_{ij}^{m} = displacement along i^{th} force due to unit force along j^{th} force, all other points being unloaded.

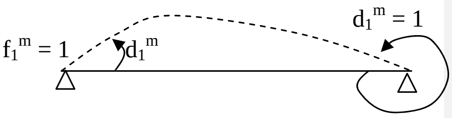
$f_1 = 1$,

Comment [SB1]: This will give the first column of F

$f_2 = 0.$

$$\begin{Bmatrix} d_1^m \\ \hline d_2^m \end{Bmatrix} \quad \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \quad \begin{Bmatrix} f_1^m \\ \hline f_2^m \end{Bmatrix}$$

F^m



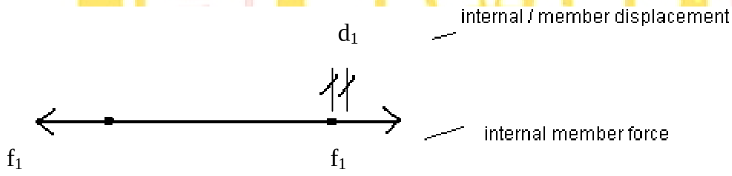
$d_1 = L/3EI = F_{11}$,
$d_2 = -L/6EI = F_{21}$
(derived from slope deflection relations)

Similarly, $f_2 = 1$ & $f_1 = 0 \Rightarrow 2^{nd}$ column of $\left[F^m\right]$

Similarly, $F_{12} = -\dfrac{L}{6EI}$ & $F_{22} = \dfrac{L}{3EI}$

$$F_m = \left(\frac{L}{6EI}\right)\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

7.4 Collective members FLEXIBILITY MATRIX FOR A TRUSS ELEMENT



$d_1 = (L/EA)f_1$

7.5 COLLECTIVE MEMBER FLEXIBILITY MATRIX OF STRUCTURE

$\{d^m\} = [F^m]\{f^m\}$ For m^{th} member.

$$\begin{Bmatrix} d^1 \\ \hline d^2 \\ \hline d^3 \\ \hline d^4 \end{Bmatrix} = \begin{bmatrix} F^1 & 0 & 0 & 0 \\ 0 & F^2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & F^N \end{bmatrix} \begin{Bmatrix} f^1 \\ \hline f^2 \\ \hline : \\ \hline f^n \end{Bmatrix}$$

Uncoupled

Matrix of all internal displacements **Matrix of all internal forces**

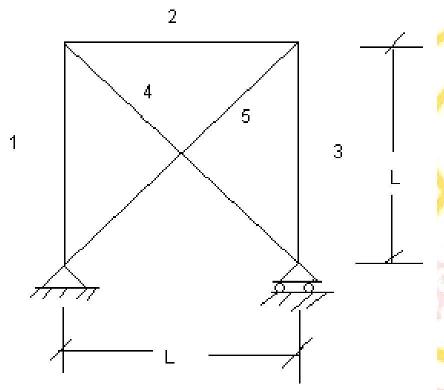
$$\{d\} = [F]_m\{f\}$$

COLLECTIVE flexibility matrix

Internal forces
[entire str.]

EXAMPLE :-

b + r = 5+ 3 = 8 **2j = 2x4 =8**



$[F^1] = [F^2] = [F^3] = \left(\dfrac{L}{EA}\right)$

$[F^4] = [F^5] = \left[\dfrac{\sqrt{2}}{EA}L\right]$

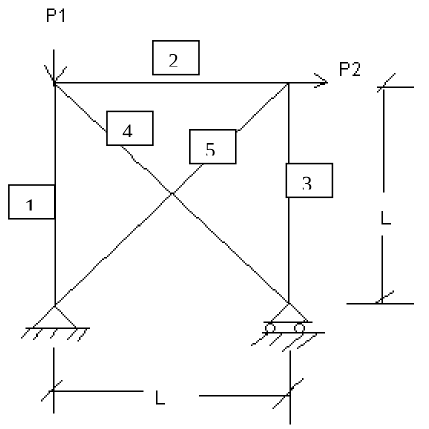
Uncoupled flexibility matrix = $[F] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \left(\dfrac{L}{EA}\right)$ $\{d\} = [F]\{f\}$

$$\{d\} = \begin{Bmatrix} d_1^1 \\ d_1^2 \\ d_1^3 \\ d_1^4 \\ d_1^5 \end{Bmatrix} \qquad \{f\} = \begin{Bmatrix} f_1^1 \\ f_1^2 \\ f_1^3 \\ f_1^4 \\ f_1^5 \end{Bmatrix}$$

7.6 TRANSFORMATION OF FORCE (DETERMINATE STRUCTURE)

To find relationship between internal forces and externally applied forces.

Let $\{P\} = \begin{Bmatrix} P_1 \\ P_2 \\ \cdots \\ P_n \end{Bmatrix}$ = Loads applied externally on structure.



How internal forces are related to {P}

$P = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$

{f} = [b] {P}

Comment [SB2]: Matrix of all internal forces

$$\begin{Bmatrix} f_1^1 \\ f_1^2 \\ f_1^3 \\ f_1^4 \\ f_1^5 \end{Bmatrix} = [b]\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

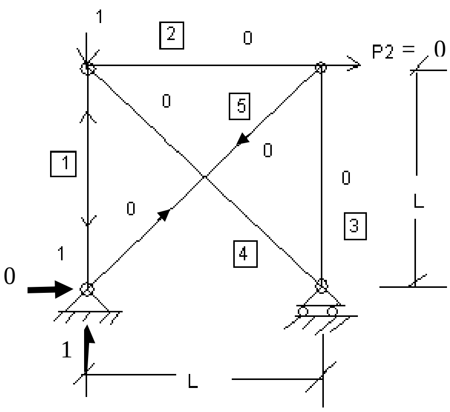
5x2 matrix

b_{ij} = Internal force f_i caused by unit external force Pj, with all other external forces = 0.

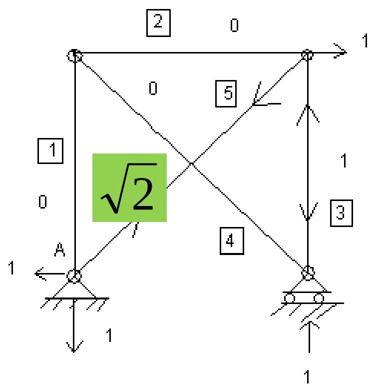
$P_1 = 1$ $P_2 = 0$ $\Rightarrow$ f_i ' s will be first column of [b]

$P_1 = 0$ $P_2 = 1$ $\Rightarrow$ f_i 's will be second column of [b]

$P_1 = 1$ $P_2 = 0$



$P_1 = 0$ $P_2 = 1$



Apply one by one

$\sum F_x = 0$

$\sum M_A = 0$

$\sum F_Y = 0$

[b] =

-1	0
0	0
0	-1
0	0
0	$\sqrt{2}$

Collection of flexibility
Matrics of member

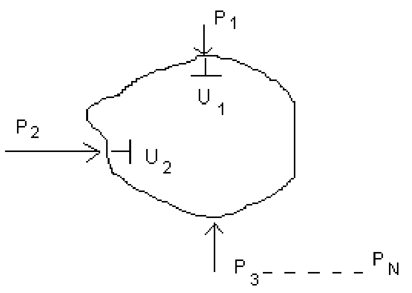
So far $\{d\} = [F]_c \{f\}$
$\{f\} = [b] \{P\}$

7.7 RELATIONSHIP BETWEEN [P] and [u]

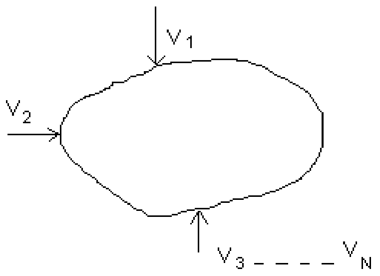
[P] = external loads / forces.
[u] = external displacement.

We will use the principle of virtual work to derive general relation.

<u>Principle of virtual work</u>



Let U_1, U_2 . . . . U_N = displacements.

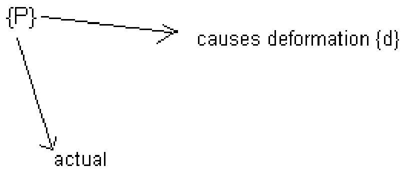


System of virtual forces only.

{V}<< {P}

Both {P} and {V} systems are existing simultaneously.
When {P} is applied, {V} rides along the displacements and performs virtual works.

External virtual work By {V} system = Internal virtual works by internal forces generated by {V}



External virtual works = $\{V\}^T\{u\}$

{v} ⟶ internal forces = [b] $\{V\} = \{f\}$ $(\{\delta f\}$ = internal forces)

Therefore internal virtual works= $\{f\}^T\{d\}$

External virtual works = internal virtual works. frd

$\{V\}^T\{u\} = \{\delta f\}^T\{d\}$

$=[[b]\{V\}]^T\{d\}$

$\{u\} = [b]^T\{d\}$ {d}----- $[F]_c$ {f}

$\{u\} = [b]^T[F]_c\{f\}$ $\{f\} - - - - - [b]\{P\}$

Therefore, $\{u\} = ([b]^T[F]_c[b])\{P\}$

F_{TS} = Total structural flexibility matrix.

For the truss structure considered earlier,

$$[F_{TS}] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \left(\frac{L}{EA}\right) \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \sqrt{2} \end{bmatrix} \left(\frac{L}{EA}\right) \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \left(\frac{L}{EA}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1+2\sqrt{2} \end{bmatrix}$$

Therefore, $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [F]_{TS} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \left(\frac{L}{EA}\right) \begin{bmatrix} 1 & 0 \\ 0 & 3.83 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

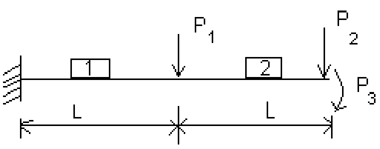
Therefore, $u_1 = \dfrac{P_1 L}{EA}$ $\quad u_2 = \dfrac{3.83 P_2}{EA}$

7.8 PROCEDURE FOR ANALYSIS OF STATICALLY DETERMINATE STRUCTURES

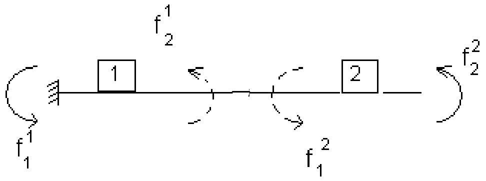
Analysis means to determine the internal forces and the deflections

1. Determine {P}
2. Define f^m, $[F^m]$, $[F]_c$
3. Form [b]
4. internal forces {f}=[b]{P}
5. $[F]_{TS} = [b]^T [F]_c [b]$
6. $\{u\} = [F]_{TS} \{P\}$

7.9 EXAMPLE 2



$$\{P\} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \qquad \{u\} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$



$$\{f\} = \begin{bmatrix} f_1^1 \\ f_2^1 \\ \\ f_1^2 \\ f_2^2 \end{bmatrix} \qquad \{d\} = \begin{bmatrix} d_1^1 \\ d_2^1 \\ \\ d_1^2 \\ d_2^2 \end{bmatrix}$$

$$[F]_c = \left(\frac{L}{6EI}\right)\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$\{f\} = [b]\{P\}$

{f}----- 4x1

[b]-------4x3

{P}-------3x1

P_1= 1.0	P_2= 1.0	P_3= 1.0

--------- All others = 0

$$[b] = \begin{bmatrix} L & 2L & 1 \\ 0 & -L & -1 \\ 0 & L & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

No. of columns = No. of P force.

$$[F]_{TS} = [b]^T [F]_c [b]$$

$$= \left(\frac{L}{6EI}\right)\begin{bmatrix} L & 0 & 0 & 0 \\ 2L & -L & L & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} L & 2L & 1 \\ 0 & -L & -1 \\ 0 & L & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \left(\frac{L}{6EI}\right)\begin{bmatrix} L & 0 & 0 & 0 \\ 2L & -L & L & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2L & 5L & 3 \\ -L & -4L & -3 \\ 0 & 2L & 3 \\ 0 & -L & -3 \end{bmatrix}$$

$$=\left(\frac{L}{6EI}\right)\begin{bmatrix} 2L^2 & 5L^2 & 3L \\ 5L^2 & 16L^2 & 12L \\ 3L & 12L & 12 \end{bmatrix}$$

Therefore, $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \left(\frac{L}{6EI}\right)\begin{bmatrix} 2L^2 & 5L^2 & 3L \\ 5L^2 & 16L^2 & 12L \\ 3L & 12L & 12 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$

Therefore, $u_1 = \frac{\left(2L^2P_1 + 5L^2P_2 + 3LP_3\right)}{6EI}$

$$u_2 = \frac{5L^3P_1 + 16L^3P_2 + 12LP_3}{6EI}$$

$$u_3 = \frac{3L^2P_1 + 12L^2P_2 + 12LP_3}{6EI}$$

If $P_2 = P_3 = 0 \Rightarrow u_1 = \frac{2P_1L^3}{6EI}$ $u_2 = \frac{5P_1L^3}{6EI}, u_3 = \frac{3P_1L^2}{6EI}$

If $P_1 = P_3 = 0$ $P_2 = P$

If the points where displacements are desired are not loaded, we must apply a fictitious load of zero value at those points.

7.10 ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES

INDETERMINATE STRUCTURES: - convert into a “primary” structure by eliminating redundant forces.

Original structure = applied loads + Unknown redundant

$f_1 = b_{11}P_1 + b_{12}P_2 + ... + b_{1N}P_n + b_{1(N+1)}x_1 + b_{1(N+2)}x_2 + ...$

$f_2 = ... + ...$

Hence, $\{f\} = [b_p \mid b_x]\begin{bmatrix} P \\ X \end{bmatrix}$

P----- Applied loads

X----- Redundant

Similarly, Displacement Vectors = $\begin{bmatrix} u_p \\ u_x \end{bmatrix}$

$u_p \rightarrow$ **Unknown** Displacements

$u_x \rightarrow$ **Prescribed** displacement (@ reaction point)

At the points of redundant –

Compatibility condition-- Displacements due to {P} + Displacements due to {X} = $\{u_x\}$

For a structure on rigid supports, u_x =0

As before-

$$\{u\} = [b]^T [F]c[b]\{P\}$$

$$= [b_p|b_x]^T[F]_c\, [b_p|b_x]\begin{Bmatrix} P \\ X \end{Bmatrix}$$

$$= \begin{bmatrix} b_p^T \\ b_x^T \end{bmatrix} \begin{bmatrix} F_c b_p & F_c b_x \end{bmatrix} \begin{Bmatrix} P \\ X \end{Bmatrix}$$

$$\therefore \begin{bmatrix} u_p \\ u_x \end{bmatrix} = \begin{bmatrix} b_p^T F_c b_p & b_p^T F_c b_x \\ b_x^T F_c b_p & b_x^T F_c b_x \end{bmatrix} \begin{bmatrix} P \\ X \end{bmatrix}$$

$$\begin{bmatrix} u_p \\ u_x \end{bmatrix} = \begin{bmatrix} F_{pp} & F_{px} \\ F_{xp} & F_{xx} \end{bmatrix} \begin{bmatrix} P \\ X \end{bmatrix}$$

If $\{u_x\}$ =0: 0= $[F_{xp}]\{P\} + [F_{xx}]\{x\}$

$\Rightarrow \{x\} = -[F_{xx}]^{-1}[F_{xp}]\{P\}$

$\{f\} = [b_p]\{P\} + [b_x]\{X\}$

$\downarrow$

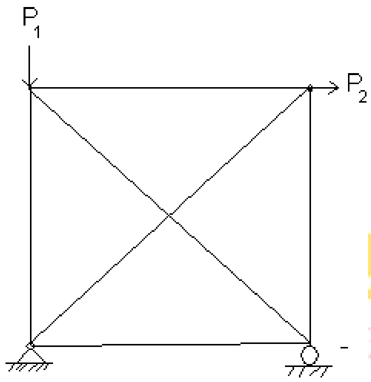
Unknown Displacements $\{u_p\} = [F_{pp}]\{P\} + [F_{px}]\{X\}$

Otherwise $\{X\} = [F_{xx}]^{-1}(\{u_x\} - [F_{xp}]\{P\})$

Summarized Procedure for Analysis of Statically Indeterminate Structures

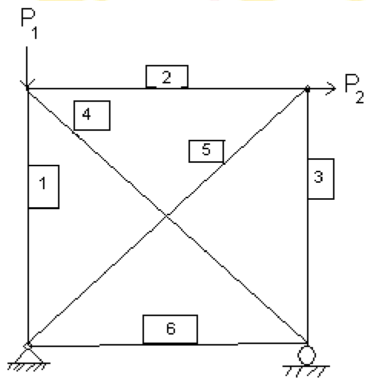
1. Define {P},{X} (DSI need to be found)
2. Define {f},from $[F]_c$
3. $[b] = [b_p|b_x]$
4. $[F_{pp}] = b_p^T F_c b_c$, $[F_{px}] = b_p^T F_c b_x$, $[F_{xp}]=[F_{px}]^T$, $[F_{xx}]=b_x^T F b_x$
5. $\{X\} = [F_{xx}]^{-1}(\{u_x\} - [F_{xp}]\{P\})$
6. $\{f\} = [b_p \mid b_x]\begin{Bmatrix} P \\ X \end{Bmatrix}$
7. $\{u_p\} = [F_{pp}]\{P\} + [F_{px}]\{X\}$

7.11 EXAMPLE 3



b+r = 9 2j=8

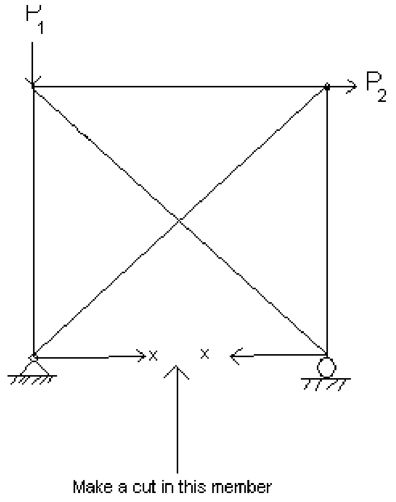
DSI=1

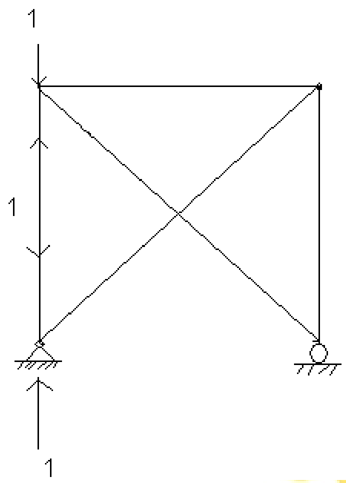


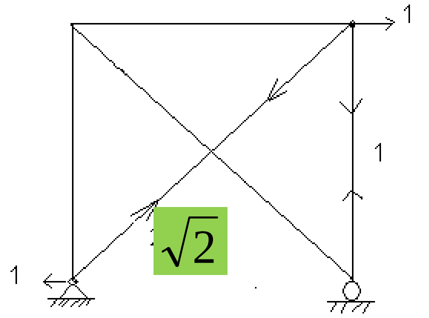
P_1=1	P_2=1	X=1

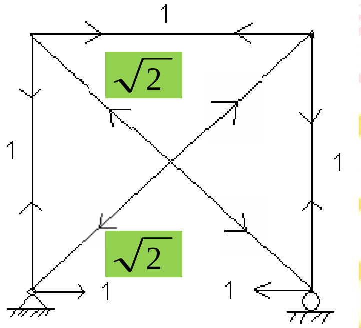
$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Primary Structure









$$[F]_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left(\frac{L}{EA}\right)$$

$$[F_{pp}] = [b_p]^T [F]_c [b_p]$$
$$[F_{px}] = [b_p]^T [F]_c [b_x]$$
$$[F_{xp}] = [F_{px}]^T = [b_x]^T [F]_c [b_p]$$
$$[F_{xx}] = [b_x]^T [F]_c [b_x]$$

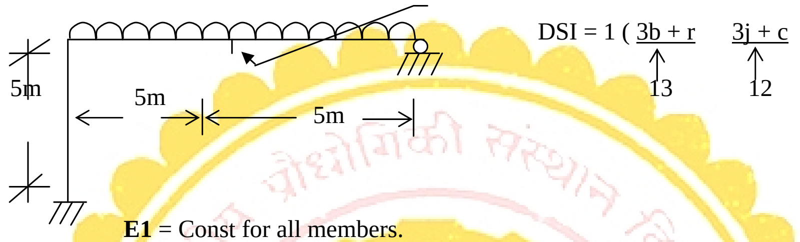
$\{u_x\}$=0 = Relative Displacements between the cut ends.

$$\therefore \{X\} = -[F_{xx}]^{-1}[F_{xp}]\{P\}$$

Ans. $\lfloor u_p \rfloor = \lfloor F_{pp} \rfloor \{P\} + \lfloor F_{px} \rfloor \{X\}$

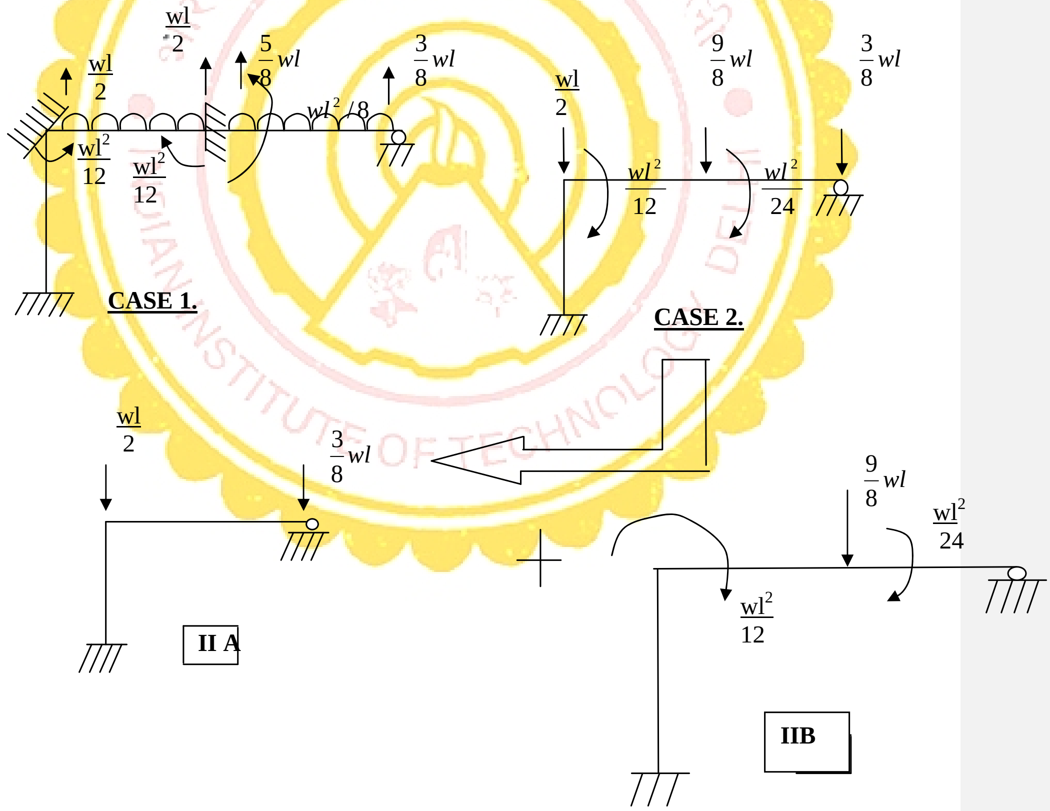
$$\{f\} = [b_p / b_x] \begin{bmatrix} P \\ X \end{bmatrix}$$

7.12 EXAMPLE 4



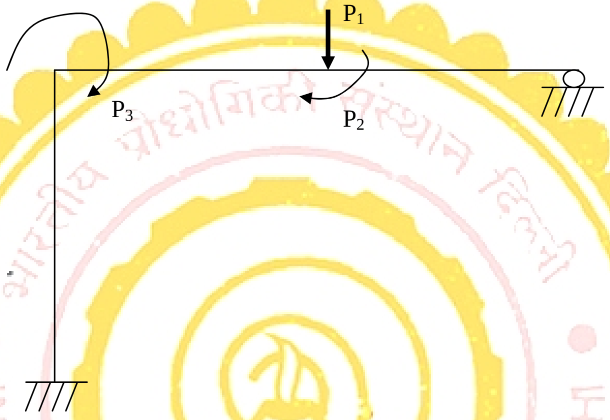
DSI = 1 (3b + r   3j + c
13   12

E1 = Const for all members.

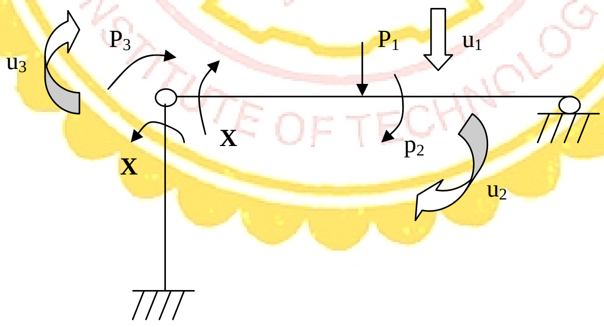


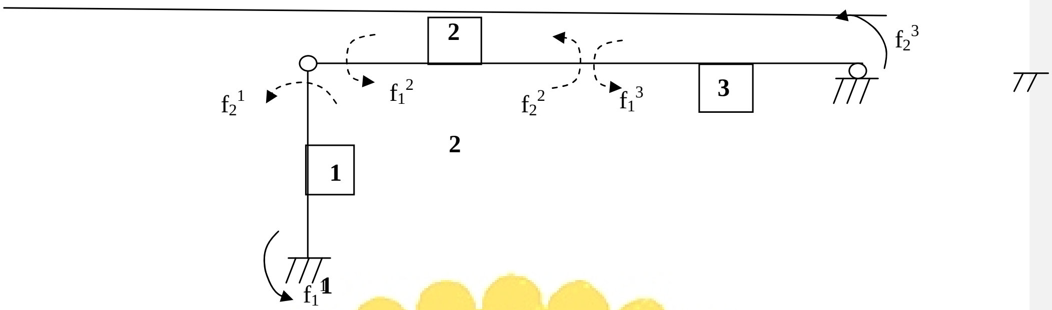
Case II A need not be analyzed.

CASE II B is equivalent to -



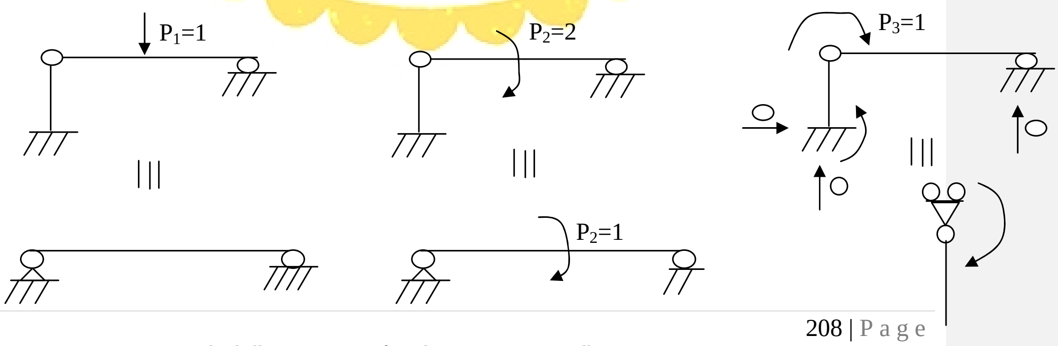
Primary structure -

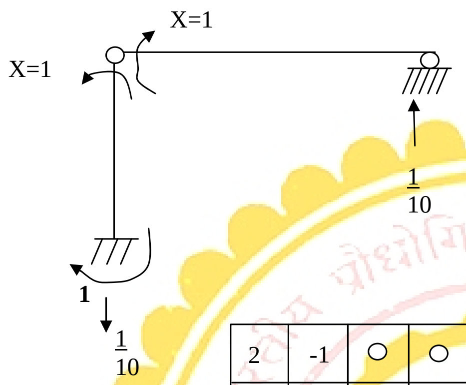


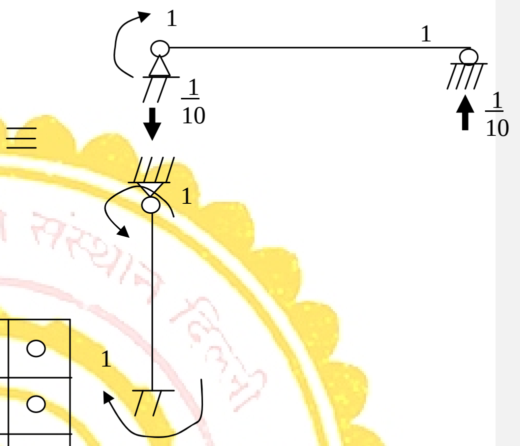


$[b]$ =

P₁ = 1	P₂ = 1	P₃ = 1	X = 1	
0	0	**1**	**-1**	f_1^1
0	0	**-1**	**1**	f_2^1
0	0	0	- 1	f_1^2
0	-0.5	0	$-\frac{1}{2}$	f_2^2
2.5	-0.5	0	$-\frac{1}{2}$	f_1^3
- 2.5	0	0	0	f_2^2







$[F]_C = \left(\frac{5}{EI}\right)$

2	-1	O	O	O	O
-1	2	O	O	O	O
O	O	2	-1	O	O
O	O	-1	2	O	O
O	O	O	O	2	-1
O	O	O	O	-1	2

$$[F]_{TS} = [b]^t [F]_c [b] = \begin{bmatrix} F_{pp} & F_{px} \\ F_{xp} & F_{xx} \end{bmatrix}$$

u_x = Rel. rotation of the members at joint B = o

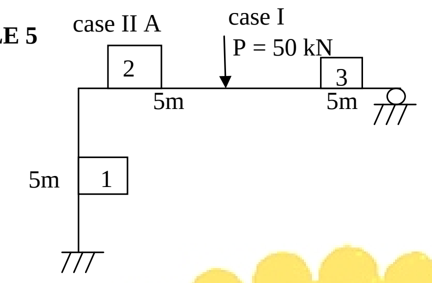
$$\begin{bmatrix} u_p \\ u_x \end{bmatrix} = \begin{bmatrix} F_{pp} & F_{px} \\ F_{xp} & F_{xx} \end{bmatrix} \begin{bmatrix} P \\ X \end{bmatrix}$$

$u_x = 0 = F_{xp}\ p + F_{xx}\ X \Rightarrow X = -\ F_{xx}^{-1}\ F_{xp}\ P$

$u_p = F_{pp}\ p + F_{px}\ X$

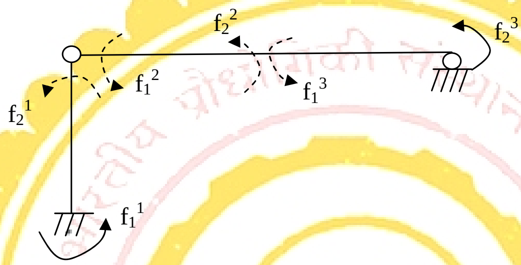
$$\{f\} = [\quad]\left[\frac{P}{X}\right] + \textbf{FEM}$$

7.13 EXAMPLE 5



$3b + r$	$3j + c$
13	12

DSI = 1



X = Same as Ex. 4

O	- 1
O	1
O	-1
$\frac{L}{2}$	$\frac{1}{2}$
- L/2	- $\frac{1}{2}$
O	O

F_c = Same as Ex. 1

$F_{PP} = b_p^T \; F_c \; bp = \frac{L^3}{EI}$ $\quad$ $F_{px} = b_p^T \; F_c \; b_x = \frac{L^2}{EI}$

$F_{xp} = b_x^T \; F_c \; bp = \frac{3L^2}{EI}$ $\quad$ $F_{xx} = b_x^T \; F_c \; b_x = \frac{3.5L}{EI}$

$$X = F_{xx}^{-1} \; F_{xp} \; P = 0.857 \; PL$$

$$U_p = F_{pp} \; P + F_{px} \; X = \frac{1.428PL^3}{EI}$$