1

LEGENDRE'S FUNCTIONS

LEGENDRE'S EQUATION

The differential equation →

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{— (1)}$$

Solⁿ of Legendre's equation is known as Legendre's function.

Let us consider Soln of eqⁿ (1) is in the form of descending power of x in series is $y = x^m(a_0 + a_1x^{-1} + a_2x^{-2} + \dots)$

[It may be ascending power of x]

Case I

When $m = n$ then

$$y_1 = a_0\left[x^n - \frac{n(n-1)}{(2n-1)\times 2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)\times 2\times 4}x^{n-4} + \dots\right]$$

→ ~~which is known~~ as is the soln of (1)

Case II

$\boxed{\text{When } m = -(n+1)}$, we have

$$y_2 = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\times(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\times 4(2n+3)(2n+5)}x^{-n-5} + \dots\right]$$

gives another solution of (1)

Note: As per your need ~~you~~ consider ascending power of x by taking

$$y = a_0x^k + a_1x^{k+1} + \dots = \sum_{r=0}^{\infty} a_r x^{k+r}$$

① **Legendre's Polynomial $P_n(x)$**

in case I y_1 if $a_0 = \frac{1\cdot 3\cdot 5 \dots (2n-1)}{\lfloor n}$ where a_0 is arbitrary constant and n is a +ve integer

thus $P_n(x) = \frac{1\cdot 3\cdot 5 \dots (2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{(2n-1)\times 2}x^{n-2} + \dots\right]$

which is a terminating series,

which is known as Legendre's function of the first kind.

Note $P_n(x) = 1$ when $x = 1$

when n is even thus total number of terms in $P_n(x)$ is $\frac{n}{2}+1$

and then $P_n(x) = \frac{1\cdot 3\cdot 5 \dots (2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{(2n-1)\times 2}x^{n-2} + \dots \frac{(-1)^{\frac{n}{2}} n(n-1)\dots \times 1}{(2n-1)(2n-2)\dots(n+1)\times 2\times 4\times \dots n}\right]$

last term

when n is odd, thus total number of terms in $P_n(x)$ is $\left(\frac{n+1}{2}\right)$

and

$P_n(x) = \frac{1\times 3\times 5 \dots (2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{(2n-1)\times 2}x^{n-2} + \dots + \frac{(-1)^{\frac{1}{2}(n-1)} n(n-1)(n-2)\times\dots 3\times 2\times x}{(2n-1)(2n-2)\dots(n+2)\times 2\times 4 \dots (n-1)}\right]$

$= \frac{1\times 3\times 5 \dots (2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{(2n-1)\times 2}x^{n-2} + \dots \frac{(-1)^{\frac{1}{2}(n-1)}\times n(n-1)(n-2)\times 3\times 2\times x}{(2n-1)(2n-2)\dots(n+2)\times 2\times 4 \dots (n-1)}\right]$

Scanned with CamScanner

Note:— Legendre function of first kind is known as Legendre Polynomial

In above discussion
case II when $m = -(n+1)$

$$y_2 = a_0\left[1x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)}x^{-n-5} + \cdots\right]$$

If we take $a_0 = \frac{\lfloor n}{1\times3\times5\cdots(2n-1)}$

$$\boxed{Q_n(x) = y_2 = \frac{\lfloor n}{1\times3\times5\cdots(2n-1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\times4\,(2n+3)(2n+5)}x^{-n-5} + \cdots\right]}$$

is Legendre function of 2nd kind denoted by $Q_n(x)$

which is nonterminating series.

General solution of Legendre's equation is

$$A P_n(x) + B Q_n(x)$$

$B\,Q_n(x)$ → Legendre function of second kind.

$P_n(x)$ → Legendre function of first kind = Legendre Polynomial

and A and B are arbitrary constants.

$\boxed{\text{RODRIGUE'S FORMULA}}$ → Legendre Polynomial

$$\boxed{P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n}$$

(4)

Proof/ $P_n(x)$ = Let $v = (x^2-1)^n$

$$\frac{dv}{dx} = n(x^2-1)^{n-1}(2x) \quad -(1)$$

Multiplying both side by (x^2-1), we get

$$(x^2-1)\frac{dv}{dx} = 2n(x^2-1)^n x$$

$$(x^2-1)\frac{dv}{dx} = 2nvx \quad - \quad (2)$$

$$\Rightarrow (x^2-1)v_1 = 2nvx$$

Now differentiating (2) by Lebnitz's Rule that is by $(n+1)$ times

$$\boxed{(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_n u_0 v_n}$$
Lebnitz's formula.

$$\{(x^2-1)v_1\}_{n+1} = 2n[vx]_{n+1}$$

$$\Rightarrow {}^{(n+1)}C_0 V_{n+2}(x^2-1) + {}^{(n+1)}C_1 V_{n+1} \times 2x + {}^{(n+1)}C_2 V_n \times 2 + {}^{(n+1)}C_3 V_{n-1} \times 0$$
$$= 2n[{}^{(n+1)}C_0 V_{n+1} x + {}^{(n+1)}C_1 V_n \times 1 + {}^{(n+1)}C_2 V_{n-1} \times 0]$$

we

[formula use ${}^nC_0 = 1,\ {}^nC_1 = n$]

$$\Rightarrow \cancel{(n+1)}\ V_{n+2}(x^2-1) + (n+1)V_{n+1} \times 2x + \frac{\lfloor n+1}{\lfloor 2 \lfloor n+1-2} V_n \times 2$$

$$\frac{\lfloor n+1}{\lfloor 2 \lfloor n-1} = \frac{(n+1)n\lfloor n-1}{\lfloor n-1 \times 2}$$

$$= 2n\left[V_{n+1}x + (n+1)V_n\right]$$

$$\Rightarrow V_{n+2}(x^2-1) + 2x(n+1)V_{n+1} + \frac{n(n+1)}{\cancel{2}} V_n \times \cancel{2}$$

$$= 2nxV_{n+1} + 2n(n+1)V_n$$

$$\Rightarrow V_{n+2}(x^2-1) + 2xV_{n+1}(n+1-n) + n(n+1)V_n[1-2] = 0$$

$$\Rightarrow V_{n+2}(x^2-1) + 2xV_{n+1} - n(n+1)V_n = 0$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\frac{d^{n+1}v}{dx^{n+1}} - n(n+1)\frac{d^n v}{dx^n} = 0 \quad - (3)$$

If we put $\frac{d^n v}{dx^n} = y$ thus

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$$

which is a Legendre's equation / $\therefore\ C\frac{d^n v}{dx^n} = P_n(x)$

$y = \frac{d^n v}{dx^n}$ is a solution of Lg eq^n

(5) where c is the constant

by but $v=(x^2-1)^n = (x+1)^n (x-1)^n$ {Leibnitz's rule

$\frac{d^n v}{dx^n} =$

$$\frac{d^n v}{dx^n} = {}^nC_0 (x+1)^n \frac{d^n}{dx^n}(x-1)^n + {}^nC_1\, n(x+1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x-1)^n$$

$$+ {}^nC_2\, n(n-1)(x+1)^{n-2} \frac{d^{n-2}}{dx^{n-2}}(x-1)^n$$

$$+ \cdots + {}^nC_n \lfloor n \,(x+1)^{n-n} (x-1)^n$$

$$+ {}^nC_n \frac{d^n}{dx^n}(x+1)^n (x-1)^n \quad \text{--- (4)}$$

When $x=1$ then all terms vanished except first term

thus

$$\frac{d^n v}{dx^n} = 2^n \lfloor n \quad \text{— Put in (3)}$$

$\because P_n(1) = 1$

thus $c\, 2^n \lfloor n = P_n(1)$

$$\Rightarrow c = \frac{1}{2^n \lfloor n}$$

Put in eq^n (3) $P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n v}{dx^n}$

$$\boxed{P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}(x^2-1)^n}$$

Ex→1 Let $P_n(x)$ be the Legendre Polynomial of degree n. Show that for any function $f(x)$, for which the nth derivative is continuous,

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^n}{2^n \lfloor n} \int_{-1}^{1} (x^2-1)^n f^n(x)\, dx$$

Soln:— $$\int_{-1}^{1} f(x) P_n(x)\, dx = \int_{-1}^{1} f(x) \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}(x^2-1)^n\, dx$$

(6)

$$= \frac{1}{2^n \lfloor n} \int_{-1}^{1} f(x) \cdot \frac{d^n}{dx^n}(x^2-1)^n \, dx$$

integrating by parts, we get

$$= \frac{1}{2^n \lfloor n}\left[\underbrace{f(x)\cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n}_{0} - \int f'(x) \times \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \, dx\right]_{-1}^{1}$$

$$= \frac{1}{2^n\lfloor n}(-1)\int_{-1}^{1} f'(x)\,\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \, dx$$

Again integrating by parts

$$= \frac{(-1)^2}{2^n \lfloor n}\int_{-1}^{1} f''(x)\, \frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n \, dx$$

Similarly integrating (n-2) times

$$\int_{-1}^{1} f(x)\,P_n(x) = \frac{(-1)^{n-2}}{2^n\lfloor n}\int_{-1}^{1} f^n(x)\,(x^2-1)^n \, dx$$

$$\boxed{= \frac{(-1)^n}{2^n \lfloor n}\int_{-1}^{1} f^n(x)\,(x^2-1)^n \, dx}$$

$(-1)^2 = (-1)^4$
2 step after same value
$(-1)^1 = (-1)^3$

Legendre's Polynomials :-

$$P_n(x) = \frac{1}{2^n\lfloor n}\,\frac{d^n}{dx^n}(x^2-1)^n \quad \text{Rodrigue's formu}$$

If $n=0$ $\quad P_0(x) = \frac{1}{2^0 \lfloor 0} = 1$

$n=1$ $\quad P_1(x) = \frac{1}{2^1\lfloor 1}\,\frac{d}{dx}(x^2-1) = \frac{1}{2}\times 2x = x$

If $n=2$ $\quad P_2(x) = \frac{1}{2^2\lfloor 2}\,\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\,\frac{d}{dx}\left[2(x^2-1)\times 2x\right]$

$$= \cancel{\frac{1}{2}\frac{d}{dx}(x^3} \quad \frac{1}{2}\left[(x^2-1)\times 1 + 2x^2\right] = \frac{1}{2}(3x^2 -$$

$$\Rightarrow \boxed{P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r \lfloor(2n-2r)}{2^n \lfloor r \, \lfloor n-r \, \lfloor n-2r}\, x^{n-2r}}$$

if n is even put $r = \frac{n}{2}$ | n is odd $r = \frac{1}{2}(n-1)$