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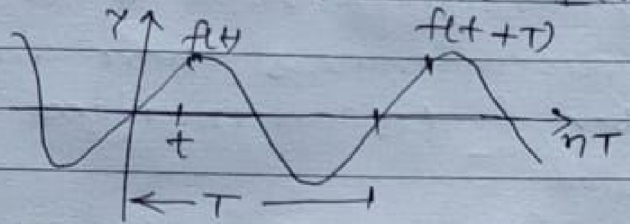
Fourier series

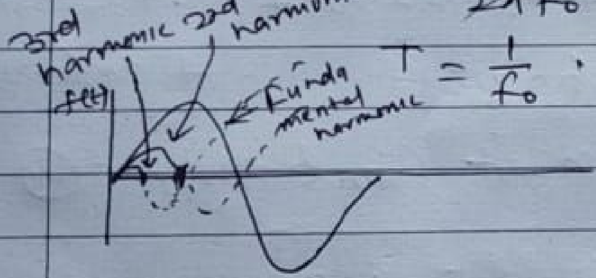
A Periodic function is that which repeats itself after regular interval of time. A function $f(t)$ is said to be periodic



$$Y = f(t)$$
$$= f(t+T)$$
$$f(t) = f(t+nT) \quad \text{where } n = 1, 2, 3 \ldots n$$

The smallest value of T is called the Period. The function $\sin w_0 t$ & $\cos w_0 t$ are commonly used periodic function.

$$T = \frac{2\pi}{w_0}$$
$$T = \frac{2\pi}{2\pi f_0}$$
$$T = \frac{1}{f_0}.$$



$$f(t) = a_0 + a_1 \cos w_0 t + a_2 \cos 2 w_0 t + \cdots + a_n \cos n w_0 t$$
$$+ b_1 \sin w_0 t + b_2 \sin 2 w_0 t + \cdots + b_n \sin n w_0 t \quad — (1)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n w_0 t + b_n \sin n w_0 t \right) \quad — (2)$$

eqn (2) This series is known as Fourier series.

Fourier series : The function $f(t)$ satisfies the following condⁿ.

(1) It is a single-value function.

(2) The integral $\int_{t_1}^{t_1+T} |f(t)|\,dt$ exists

for any arbitrary value of t_1.

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(3) If $f(t)$ is discontinuous there are finite number of discontinuities in any one period.

(4) The function $f(t)$ has a finite number of Maxima & Minima in any one period.

These condition are called

Dirichlet condition:

①, ② & ④ are strong Dirichlet condⁿ but ② is weak Dirichlet condⁿ.

periodic function $f(t)$ of period T satisfying Dirichlet condition can be expanded into the series

$$f(t) = a_0 + a_1 \cos w_0 t + a_2 \cos 2w_0 t + \dots + a_n \cos n w_0 t + b_1 \sin n w_0 t + b_2 \sin n w_0 t + \dots + b_n \sin n w_0 t \quad \text{①}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n w_0 t + b_n \sin n w_0 t) \quad \text{②}$$

constt a_0, a_n & b_n are called Fourier series coefficient constt. of $f(t)$

The sine & cosine terms are called harmonics.

n represents the order of harmonics

when $n=1$ is called 1st harmonics

$n=2$ " " 2nd harmonics.

The term $(a_n \cos n w_0 t + b_n \sin n w_0 t)$ is called the nth harmonics

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Some useful integrals:

(1) $\int_{t_1}^{t_1+T} \sin n\omega_0 t\, dt = 0$, $\int_0^T \sin n\omega_0 t\, dt = 0$.

$\int_{-T/2}^{T/2} \sin n\omega_0 t\, dt = 0$, $\int_{-T/4}^{3T/4} \sin n\omega_0 t\, dt = 0$

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$\int_0^T \sin n\omega_0 t\, dt = 0$, $\int_0^{2\pi} \sin n\theta\, d\theta = 0$

(2) $\int_{t_1}^{t_1+T} \cos n\omega_0 t\, dt = 0$, $\int_0^{2\pi} \cos n\theta\, d\theta = 0$.

(3) $\int_{t_1}^{t_1+T} \sin m\omega_0 t \cdot \cos n\omega_0 t = 0$ for all m & n

Proof: $\frac{1}{2}\int_{t_1}^{t_1+T} 2\sin m\omega_0 t \cdot \cos n\omega_0 t\, dt$

$= \frac{1}{2}\int_{t_1}^{t_1+T} [\sin (m+n)\omega_0 t + \sin (m-n)\omega_0 t]\, dt$

$= \frac{1}{2}\left[\int_{t_1}^{t_1+T} \sin (m+n)\omega_0 t\, dt + \int_{t_1}^{t_1+T} \sin (m-n)\omega_0 t\, dt\right]$

$= \frac{0+0}{2} = 0.$

(4) $\int_0^{2\pi} \sin m\theta \cdot \cos n\theta = 0$ for all m & n

(5) $\int_0^{2\pi} \sin^2 n\theta\, d\theta = \pi$

Proof: $= \frac{1}{2}\int_0^{2\pi} \sin^2 n\theta\, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (1 - \cos 2n\theta)\, d\theta$

$= \frac{1}{2}\left[\int_0^{2\pi} d\theta - \int_0^{2\pi} \cos 2n\theta\, d\theta\right]$

$= \frac{1}{2}[2\pi - 0] = \frac{1}{\cancel{2}} \times \cancel{2}\pi = \pi.$

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(6) $$\int_{t_1}^{t_1+T} \sin m\omega_0 t \cdot \sin n\omega_0 t \, dt = \begin{cases} T/2 & m=n\neq 0 \\ 0 & m\neq n \\ 0 & m \text{ or } n = 0 \end{cases}$$

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Proof:

$$\frac{1}{2}\int_{t_1}^{t_1+T} 2\sin m\omega_0 t \cdot \sin n\omega_0 t \, dt$$

$$= \frac{1}{2}\int_{t_1}^{t_1+T} [\cos(m-n)\omega_0 t - \cos(m+n)\omega_0 t]\, dt$$

$$= \frac{1}{2}\left[\int_{t_1}^{t_1+T} \cos(m-n)\omega_0 t \, dt - \int_{t_1}^{t_1+T} \cos(m+n)\omega_0 t \, dt\right] \quad \text{—} (*)$$

Case ① when $m=n\neq 0$ $(\because m=n)$

$$= \frac{1}{2}\left[\int_{t_1}^{t_1+T} \cos 0 \cdot dt - \int_{t_1}^{t_1+T} \cos(m+n)\omega_0 t \, dt\right.$$

$$= \frac{1}{2}\left[\int_{t_1}^{t_1+T} dt - \int_{t_1}^{t_1+T} \cos 2m\omega_0 t \, dt\right.$$

$$= \frac{1}{2}\left[\int_{t_1}^{t_1+T} dt - 0\right]$$

$$= \frac{1}{2}\left[[t]_{t_1}^{t_1+T}\right.$$

$$= \frac{1}{2}[\cancel{t_1} + T - \cancel{t_1}] = \frac{T}{2}.$$

Case (II) when $m\neq n$ $m>n$ $(m-n)=d$ (say)

$$\frac{1}{2}\left[\int_{t_1}^{t_1+T} \cos(m-n)\omega_0 t \, dt - \int_{t_1}^{t_1+T} \cos(m+n)\omega_0 t \, dt\right.$$

$$= \frac{1}{2}\left[\int_{t_1}^{t_1+T} \cos d \cdot \omega_0 t \, dt - \int_{t_1}^{t_1+T} \cos(m+n)\omega_0 t \, dt\right.$$

$$= \frac{1}{2}[0-0] = 0.$$

Case (III) m or $n = 0$

$$\frac{1}{2}\left[\int_{t_1}^{t_1+T} \cos n\omega_0 t \, dt - \int_{t_1}^{t_1+T} \cos m\omega_0 t \, dt\right.$$

$$= \frac{1}{2}[0-0] = \underline{\underline{0}},$$

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(7) $\int_0^{2\pi} \sin m\theta \cdot \sin n\theta \, d\theta = 0$ when $m \neq n$

proof
$= \frac{1}{2}\int_0^{2\pi} 2\sin m\theta \cdot \sin n\theta \, d\theta$

$= \frac{1}{2}\left[\int_0^{2\pi}\{\cos(m-n)\theta - \cos(m+n)\theta\} d\theta\right.$

$= \frac{1}{2}\left[\int_0^{2\pi}\cos(m-n)\theta \cdot d\theta - \int_0^{2\pi}\cos(m+n)\theta \, d\theta\right.$

$m - n = \alpha$ (say)

$= \frac{1}{2}\left[\int_0^{2\pi}\cos\alpha\theta \, d\theta - \int_0^{2\pi}\cos(m+n)\theta \, d\theta\right.$

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$= \frac{0+0}{2} = 0$

(8) $\int_{t_1}^{t_1+T} \cos m w_0 t \cdot \cos n w_0 t \, dt = \begin{cases} T/2 & m = n \neq 0 \\ 0 & m \neq n \\ T & m = n. \end{cases}$

Proof
$= \frac{1}{2}\int_{t_1}^{t_1+T} 2\cos m w_0 t \cdot \cos n w_0 t \, dt$

$= \frac{1}{2}\int_{t_1}^{t_1+T} \cos(m+n)w_0 t + \cos(m-n)w_0 t \, dt$

$= \frac{1}{2}\left[\int_{t_1}^{t_1+T}\cos(m+n)w_0 t \, dt + \int_{t_1}^{t_1+T}\cos(m-n)w_0 t \, dt\right.$ — (*)

Case I. when $m = n \neq 0$ $m = n$

$\frac{1}{2}\left[\int_{t_1}^{t_1+T}\cos 2m w_0 t \, dt + \int_{t_1}^{t_1+T}\cos 0 \, w_0 t \, dt\right.$

$= \frac{1}{2}[0 + T] = \frac{T}{2}$.

Case (II) $m \neq n$. $m - n = \alpha$ (say)

$\frac{1}{2}\left[\int_{t_1}^{t_1+T}\cos(m+n)w_0 t \, dt + \int_{t_1}^{t_1+T}\cos\alpha \cdot w_0 t \cdot dt\right]$

$= \frac{1}{2}[0+0] = 0$

Case (III) $m = n = 0$

$\frac{1}{2}\left[\int_{t_1}^{t_1+T}\cos 0 \, w_0 t \, dt + \int_{t_1}^{t_1+T}\cos 0 \, w_0 t \, dt\right.$

$= \frac{1}{2}[T+T] = \frac{2T}{2} = T$.

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(9) $\int_0^{2\pi} \cos^2 n\theta \, d\theta = \pi$

proof $\frac{1}{2}\int_0^{2\pi} 2\cos^2 n\theta \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2n\theta) \, d\theta$

$= \frac{1}{2}\left[\int_0^{2\pi} d\theta + \int_0^{2\pi} \cos 2n\theta \, d\theta\right.$

$= \frac{1}{2}\left[[\theta]_0^{2\pi} + 0\right.$

$= \frac{1}{\cancel{2}} \times \cancel{2}\pi = \pi$

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(10) $e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t$

$\int_{t_1}^{t_1+T} e^{jn\omega_0 t} \, dt = \int_{t_1}^{t_1+T} (\cos n\omega_0 t + j \sin n\omega_0 t) \, dt$

$= \int_{t_1}^{t_1+T} \cos n\omega_0 t \, dt + j \int_{t_1}^{t_1+T} \sin n\omega_0 t \, dt$

$= 0 + 0 = 0$

(11) $\int_{t_1}^{t_1+T} e^{j(n-m)\omega_0 t} \, dt = \begin{cases} 0 & n \neq m \\ T & n = m \end{cases}$

proof $\int_{t_1}^{t_1+T} \cos(n-m)\omega_0 t + j \int_{t_1}^{t_1+T} \sin(n-m)\omega_0 t \, dt$

Case (I) $n \neq m$ $n - m = \alpha$ (say)

$\int_{t_1}^{t_1+T} \cos \alpha \omega_0 t \, dt + j \int_{t_1}^{t_1+T} \sin \alpha \cdot \omega_0 t \, dt$

$0 + j \times 0 = 0$

Case (II) $n = m$ $n - m = 0$

$\int_{t_1}^{t_1+T} \cos 0 \, \omega_0 t \, dt + j \int_{t_1}^{t_1+T} \sin 0 \, \omega_0 t \, dt$

$[t]_{t_1}^{t_1+T} + j \times 0$

$[\cancel{t_1} + T - \cancel{t_1}] = T$

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evaluation of a_0

Integrating both side over one period (t_1, t_1+T) where t_1 is arbitrary

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{—} ①$$

$$\int_{t_1}^{t_1+T} f(t)\,dt = \int_{t_1}^{t_1+T} a_0\,dt + \int_{t_1}^{t_1+T} \Sigma (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\,dt$$

$$\int_{t_1}^{t_1+T} f(t)\,dt = a_0 (t)\Big|_{t_1}^{t_1+T} + \sum_{n=1}^{\infty} \int_{t_1}^{t_1+T} a_n \cos n\omega_0 t\,dt + \int_{t_1}^{t_1+T} b_n \sin n\omega_0 t\,dt$$

$$\int_{t_1}^{t_1+T} f(t)\,dt = a_0 T + \sum_{n=1}^{\infty} \left[a_n \int_{t_1}^{t_1+T} \cos n\omega_0 t\,dt + b_n \int_{t_1}^{t_1+T} \sin n\omega_0 t\,dt\right.$$

$$\int_{t_1}^{t_1+T} f(t)\,dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} f(t)\,dt}$$

we know that

$$\int_{t_1}^{t_1+T} \cos n\omega_0 t\,dt = 0$$

$$\int_{t_1}^{t_1+T} \sin n\omega_0 t\,dt = 0$$

evaluation of a_n

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{—} ①$$

To evaluate a_n, the amplitude of $\cos n\omega_0 t$ component of $f(t)$, we multiply eqn ① by $\cos m\omega_0 t$ and integrate over one period (t_1, t_1+T)

$$f(t) \cos m\omega_0 = a_0 \cos m\omega_0 t + \sum_{n=1}^{\infty} a_n \cos m\omega_0 t \cdot \cos n\omega_0 t + b_n \sin n\omega_0 t \cdot \cos m\omega_0 t$$

$$\int_{t_1}^{t_1+T} f(t) \cos m\omega_0 t\,dt = \int_{t_1}^{t_1+T} a_0 \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty} \int_{t_1}^{t_1+T} a_n \cos m\omega_0 t \cdot \cos n\omega_0 t\,dt + \int_{t_1}^{t_1+T} b_n \sin n\omega_0 t \cdot \cos m\omega_0 t\,dt$$

$$\int_{t_1}^{t_1+T} f(t) \cos m\omega_0 t\,dt = a_0 \int_{t_1}^{t_1+T} \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty} a_n \int_{t_1}^{t_1+T} \cos m\omega_0 t \cdot \cos n\omega_0 t\,dt + b_n \int_{t_1}^{t_1+T} \sin n\omega_0 t \cdot \cos m\omega_0 t\,dt$$

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from formula

$$\int_{t_1}^{t_1+T} \cos m\omega_0 t\, dt = 0$$

$$\int_{t_1}^{t_1+T} \sin n\omega_0 t \cdot \cos m\omega_0 t\, dt = 0$$

$$\int_{t_1}^{t_1+T} \cos m\omega_0 t \cdot \cos n\omega_0 t\, dt = \begin{cases} 0 & m \neq n \\ \frac{T}{2} & m = n \end{cases}$$ from privious case (8)

Then, we can write,

$$\int_{t_1}^{t_1+T} f(t) \cos m\omega_0 t\, dt = \sum_{n=1}^{\infty} a_n \int_{t_1}^{t_1+T} \cos m\omega_0 t \cdot \cos n\omega_0 t\, dt$$

Case I $m \neq n$

That means

Right side part is zero.

Case II $m = n$

$$\int_{t_1}^{t_1+T} f(t) \cos m\omega_0 t\, dt = a_n \times \frac{T}{2}$$

$$\boxed{a_n = \frac{2}{T} \int_{t_1}^{t_1+T} f(t) \cos m\omega_0 t\, dt}$$

evaluation of b_n

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

Multiplying both side by $\sin m\omega_0 t$

$$f(t) \sin m\omega_0 t = a_0 \sin m\omega_0 t + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t \cdot \sin m\omega_0 t + b_n \sin n\omega_0 t \cdot \sin m\omega_0 t)$$

Integrating both side over one period (t_1, t_1+T)

$$\int_{t_1}^{t_1+T} f(t) \sin m\omega_0 t\, dt = \int_{t_1}^{t_1+T} a_0 \sin m\omega_0 t\, dt + \sum_{n=1}^{\infty} \int_{t_1}^{t_1+T} a_n \cos n\omega_0 t \sin m\omega_0 t\, dt + \int_{t_1}^{t_1+T} b_n \sin n\omega_0 t \sin m\omega_0 t\, dt$$

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We know that

$$\int_{t_1}^{t_1+T} \sin m\omega_0 t \, dt = 0 \qquad \text{from case } 1$$

$$\int_{t_1}^{t_1+T} \cos n\omega_0 t \cdot \sin m\omega_0 t \, dt = 0 \quad \text{from case } 3$$

Therefore,

$$\int_{t_1}^{t_1+T} f(t) \sin m\omega_0 t \, dt = b_n \int_{t_1}^{t_1+T} \sin n\omega_0 t \cdot \sin m\omega_0 t \, dt$$

from case 8

$$\int_{t_1}^{t_1+T} \sin m\omega_0 t \cdot \sin n\omega_0 t \, dt = \begin{cases} T/2 & m = n \neq 0 \\ 0 & m \neq n. \end{cases}$$

So, we can write.

$$\int_{t_1}^{t_1+T} \sin m\omega_0 t \cdot \sin n\omega_0 t \, dt = T/2$$

$$\int_{t_1}^{t_1+T} f(t) \sin m\omega_0 t \, dt = b_n \times \frac{T}{2}$$

$$\boxed{\therefore b_n = \frac{2}{T} \int_{t_1}^{t_1+T} f(t) \sin m\omega_0 t \, dt}$$

$$a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} f(t) \, dt.$$

$$a_n = \frac{2}{T} \int_{t_1}^{t_1+T} f(t) \cos m\omega_0 t \, dt.$$

$$b_n = \frac{2}{T} \int_{t_1}^{t_1+T} f(t) \sin m\omega_0 t \, dt.$$

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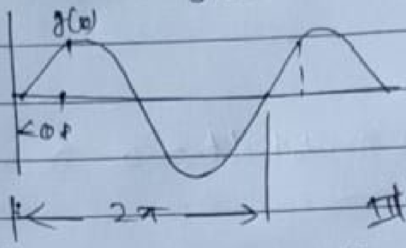
Function of period 2π

let $g(\theta)$ be a periodic function of period 2π



$f(t) = f(t+T)$

lly, $g(\theta) = g(\theta + 2\pi)$

$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$

$\theta = \omega_0 t$

similarly,

$g(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$

$\theta = 2\pi f_0 t$

$= \frac{2\pi}{T} \cdot t$

$\omega_0 = 2\pi f_0 = 2\pi \times \frac{1}{T_0} = \frac{2\pi}{T_0}$

$a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} f(t)\,dt$

$d\theta = \frac{2\pi}{T} dt$

$a_0 = \frac{1}{T} \int_0^T f(t)\,dt$

$dt = \frac{T}{2\pi} \cdot d\theta$

$f(t) = g(\theta)$

suppose $t_1 = 0$

$t_1 + T \Rightarrow T$

$0 \to T$

$-\frac{T}{2} \to \frac{T}{2}$

$-\frac{T}{4} \to \frac{3T}{4}$

$a_0 = \frac{1}{T} \int_0^{2\pi} g(\theta) \cdot \frac{T}{2\pi} \cdot d\theta$

$$\boxed{a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\,d\theta}$$

$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t\,dt$

$f(t) = g(\theta)$, $T \to 2\pi$, $\omega_0 t = \theta$

$dt = \frac{d\theta}{dt}$

$dt = \frac{T}{2\pi} d\theta$

$a_n = \frac{2}{T} \int_0^{2\pi} g(\theta) \cos n\theta \frac{T}{2\pi} d\theta$

$= \frac{\cancel{2}}{\cancel{T}} \times \frac{\cancel{T}}{\cancel{2}\pi} \int_0^{2\pi} g(\theta) \cos n\theta\,d\theta$

$$\boxed{a_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos n\theta\,d\theta}$$

$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t\,dt$

similarly

$$\boxed{b_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta\,d\theta}$$