CHAPTER 15

Beam Analysis Using the Stiffness Method

The concepts presented in the previous chapter will be extended here and applied to the analysis of beams. It will be shown that once the member stiffness matrix and the transformation matrix have been developed, the procedure for application is exactly the same as that for trusses. Special consideration will be given to cases of differential settlement and temperature.

15-1 Preliminary Remarks

Before we show how the stiffness method applies to beams, we will first discuss some preliminary concepts and definitions related to these members.

Member and Node Identification. In order to apply the stiffness method to beams, we must first determine how to subdivide the beam into its component finite elements. In general, each element must be free from load and have a prismatic cross section. For this reason the nodes of each element are located at a support or at points where members are connected together, where an external force is applied, where the cross-sectional area suddenly changes, or where the vertical or rotational displacement at a point is to be determined. For example, consider the beam in Fig. 15–1*a*. Using the same scheme as that for trusses, four nodes are specified numerically within a circle, and the three elements are identified numerically within a square. Also, notice that the "near" and "far" ends of each element are identified by the arrows written alongside each element.

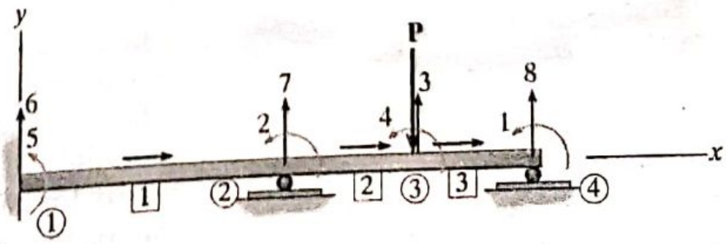
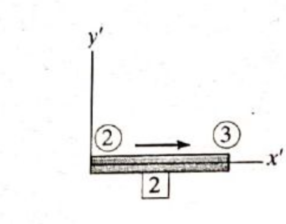


Fig. 15–1

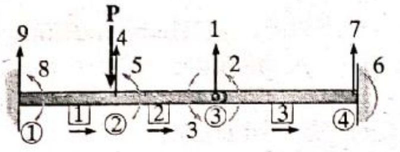
(a)

Scanned with CamScanner

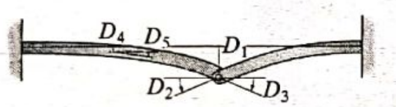


(b)

Fig. 15–1

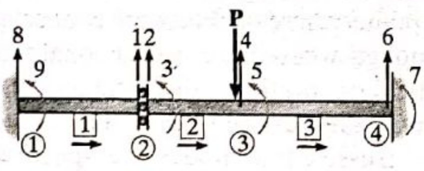


(a)

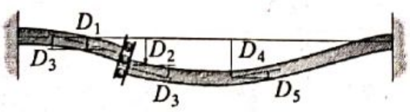


(b)

Fig. 15–2



(a)



(b)

Fig. 15–3

Global and Member Coordinates. The global coordinate system will be identified using x, y, z axes that generally have their origin at a node and are positioned so that the nodes at other points on the beam all have positive coordinates, Fig. 15–1*a*. The local or member x', y', z' coordinates have their origin at the "near" end of each element, and the positive x' axis is directed towards the "far" end. Figure 15–1*b* shows these coordinates for element 2. In both cases we have used a right-handed coordinate system, so that if the fingers of the right hand are curled from the x (x') axis towards the y (y') axis, the thumb points in the positive direction of the z (z') axis, which is directed out of the page. Notice that for each beam element the x and x' axes will be collinear and the global and member coordinates will all be parallel. Therefore, unlike the case for trusses, here we will not need to develop transformation matrices between these coordinate systems.

Kinematic Indeterminacy. Once the elements and nodes have been identified, and the global coordinate system has been established, the degrees of freedom for the beam and its kinematic determinacy can be determined. If we consider the effects of both bending and shear, then *each node* on a beam can have two degrees of freedom, namely, a vertical displacement and a rotation. As in the case of trusses, these displacements will be identified by code numbers. The lowest code numbers will be used to identify the unknown displacements (unconstrained degrees of freedom), and the highest numbers are used to identify the known displacements (constrained degrees of freedom). Recall that the reason for choosing this method of identification has to do with the convenience of later partitioning the structure stiffness matrix, so that the unknown displacements can be found in the most direct manner.

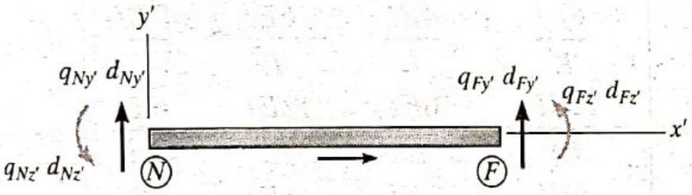
To show an example of code-number labeling, consider again the continuous beam in Fig. 15–1*a*. Here the beam is kinematically indeterminate to the fourth degree. There are eight degrees of freedom, for which code numbers 1 through 4 represent the unknown displacements, and numbers 5 through 8 represent the known displacements, which in this case are all zero. As another example, the beam in Fig. 15–2*a* can be subdivided into three elements and four nodes. In particular, notice that the internal hinge at node 3 deflects the same for both elements 2 and 3; however, the rotation at the end of each element is different. For this reason three code numbers are used to show these deflections. Here there are nine degrees of freedom, five of which are unknown, as shown in Fig. 15–2*b*, and four known; again they are all zero. Finally, consider the slider mechanism used on the beam in Fig. 15–3*a*. Here the deflection of the beam is shown in Fig. 15–3*b*, and so there are five unknown deflection components labeled with the lowest code numbers. The beam is kinematically indeterminate to the fifth degree.

Development of the stiffness method for beams follows a similar procedure as that used for trusses. First we must establish the stiffness matrix for each element, and then these matrices are combined to form the beam or structure stiffness matrix. Using the structure matrix equation, we can then proceed to determine the unknown displacements at the nodes and from this determine the reactions on the beam and the internal shear and moment at the nodes.

15–2 Beam-Member Stiffness Matrix

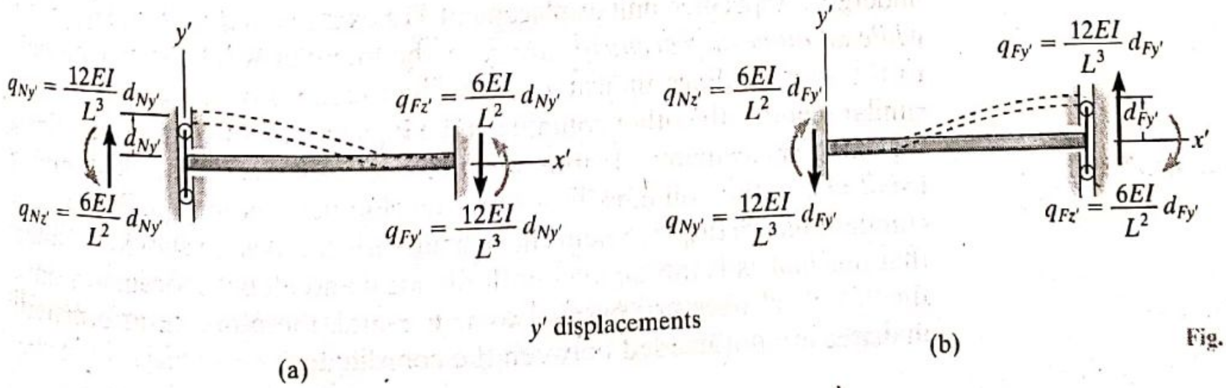
In this section we will develop the stiffness matrix for a beam element or member having a constant cross-sectional area and referenced from the local x', y', z' coordinate system, Fig. 15–4. The origin of the coordinates is placed at the "near" end N, and the positive x' axis extends toward the "far" end F. There are two reactions at each end of the element, consisting of shear forces $q_{Ny'}$ and $q_{Fy'}$ and bending moments $q_{Nz'}$ and $q_{Fz'}$. These loadings all act in the positive coordinate directions. In particular, the moments $q_{Nz'}$ and $q_{Fz'}$ are positive *counterclockwise*, since by the right-hand rule the moment vectors are then directed along the positive z' axis, which is out of the page.

Linear and angular displacements associated with these loadings also follow this same positive sign convention. We will now impose each of these displacements separately and then determine the loadings acting on the member caused by each displacement.



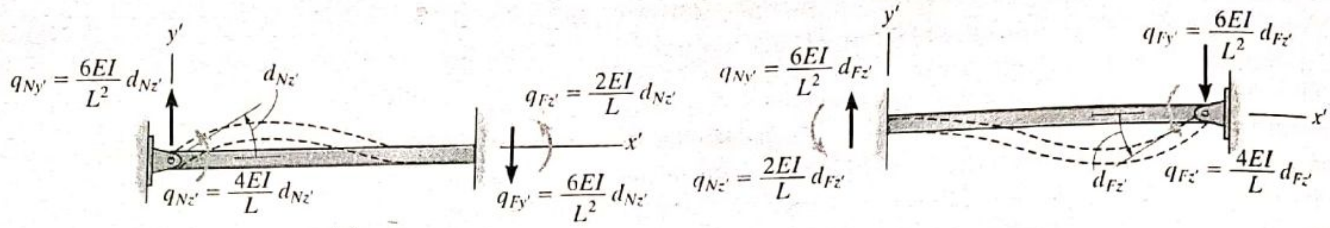
positive sign convention Fig. 15–4

y' Displacements. When a positive displacement $d_{Ny'}$ is imposed while other possible displacements are prevented, the resulting shear forces and bending moments that are created are shown in Fig. 15–5a. In particular, the moment has been developed in Sec. 11–2 as Eq. 11–5. Likewise, when $d_{Fy'}$ is imposed, the required shear forces and bending moments are given in Fig. 15–5b.



y' displacements

(a) (b) Fig. 15–5



(a)

Fig. 15–6

z' Rotations. If a positive rotation $d_{Nz'}$ is imposed while all other possible displacements are prevented, the required shear forces and moments necessary for the deformation are shown in Fig. 15–6*a*. In particular, the moment results have been developed in Sec. 11–2 as Eqs. 11–1 and 11–2. Likewise, when $d_{Fz'}$ is imposed, the resultant loadings are shown in Fig. 15–6*b*.

By superposition, if the above results in Figs. 15–5 and 15–6 are added, the resulting four load-displacement relations for the member can be expressed in matrix form as

$$
\begin{bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} \quad N_{y'} \quad & \quad N_{z'} \quad & \quad F_{y'} \quad & \quad F_{z'} \quad \end{array} \\
\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}
\begin{bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix}
\qquad (15\text{–}1)
$$

These equations can also be written in abbreviated form as

$$
\mathbf{q} = \mathbf{k}\mathbf{d} \qquad (15\text{–}2)
$$

The symmetric matrix $\mathbf{k}$ in Eq. 15–1 is referred to as the *member stiffness matrix*. The 16 influence coefficients $\mathbf{k}_{ij}$ that comprise it account for the shear-force and bending-moment displacements of the member. Physically these coefficients represent the load on the member when the member undergoes a specified unit displacement. For example, if $d_{Ny'} = 1$, Fig. 15–5*a*, *while all other displacements are zero,* the member will be subjected only to the four loadings indicated in the first column of the $\mathbf{k}$ matrix. In a similar manner, the other columns of the $\mathbf{k}$ matrix are the member loadings for unit displacements identified by the degree-of-freedom code numbers listed above the columns. From the development, both equilibrium and compatibility of displacements have been satisfied. Also, it should be noted that this matrix is the *same* in both the local and global coordinates since the x', y', z' axes are parallel to x, y, z and, therefore, transformation matrices are not needed between the coordinates.

15-3 Beam-Structure Stiffness Matrix

Once all the member stiffness matrices have been found, we must assemble them into the structure stiffness matrix **K**. This process depends on first knowing the *location* of each element in the member stiffness matrix. Here the rows and columns of each **k** matrix (Eq. 15–1) are identified by the two code numbers at the near end of the member $(N_{y'}, N_{z'})$ followed by those at the far end $(F_{y'}, F_{z'})$. Therefore, when assembling the matrices, each element must be placed in the same location of the **K** matrix. In this way, **K** will have an order that will be equal to the highest code number assigned to the beam, since this represents the total number of degrees of freedom. Also, where several members are connected to a node, their member stiffness influence coefficients will have the same position in the **K** matrix and therefore must be algebraically added together to determine the nodal stiffness influence coefficient for the structure. This is necessary since each coefficient represents the nodal resistance of the structure in a particular direction (y' or z') when a unit displacement (y' or z') occurs either at the same or at another node. For example, $\mathbf{K}_{23}$ represents the load in the direction and at the location of code number "2" when a unit displacement occurs in the direction and at the location of code number "3."

15-4 Application of the Stiffness Method for Beam Analysis

Once the structure stiffness matrix is determined, the loads at the nodes of the beam can be related to the displacements using the structure stiffness equation

$$\mathbf{Q} = \mathbf{K}\mathbf{D}$$

Here **Q** and **D** are column matrices that represent both the known and unknown loads and displacements. Partitioning the stiffness matrix into the known and unknown elements of load and displacement, we have

$$\begin{bmatrix} \mathbf{Q}_k \\ \hdashline \mathbf{Q}_u \end{bmatrix} = \left[\begin{array}{c:c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} \end{array}\right] \begin{bmatrix} \mathbf{D}_u \\ \hdashline \mathbf{D}_k \end{bmatrix}$$

which when expanded yields the two equations

$$\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u + \mathbf{K}_{12}\mathbf{D}_k \tag{15–3}$$

$$\mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u + \mathbf{K}_{22}\mathbf{D}_k \tag{15–4}$$

The unknown displacements $\mathbf{D}_u$ are determined from the first of these equations. Using these values, the support reactions $\mathbf{Q}_u$ are computed for the second equation.

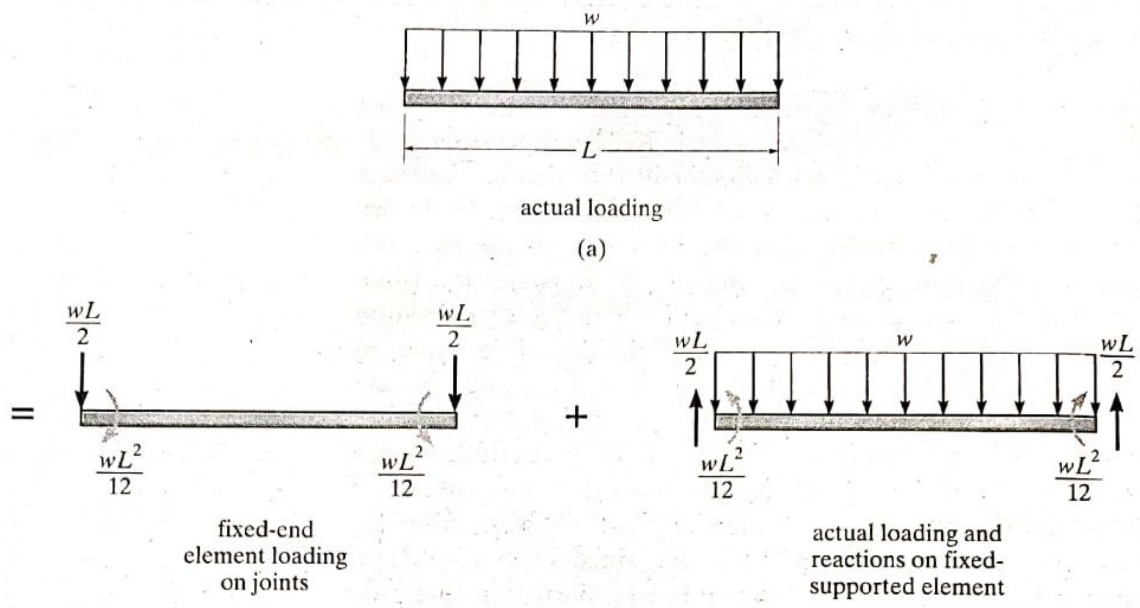


Fig. 15–7

Intermediate Loadings. For application, it is important that the elements of the beam be free of loading along its length. This is necessary since the stiffness matrix for each element was developed for loadings applied only at its ends. (See Fig. 15–4.) Oftentimes, however, beams will support a distributed loading, and this condition will require modification in order to perform the matrix analysis.

To handle this case, we will use the principle of superposition in a manner similar to that used for trusses discussed in Sec. 14–8. To show its application, consider the beam element of length L in Fig. 15–7*a*, which is subjected to the uniform distributed load w. First we will apply fixed-end moments and reactions to the element, which will be used in the stiffness method, Fig. 15–7*b*. We will refer to these loadings as a column matrix $-\mathbf{q}_0$. Then the distributed loading and its reactions $\mathbf{q}_0$ are applied, Fig. 15–7*c*. The actual loading within the beam is determined by adding these two results. The fixed-end reactions for other cases of loading are given on the inside back cover. In addition to solving problems involving lateral loadings such as this, we can also use this method to solve problems involving temperature changes or fabrication errors.

Member Forces. The shear and moment at the ends of each beam element can be determined using Eq. 15–2 and adding on any fixed-end reactions $\mathbf{q}_0$ if the element is subjected to an intermediate loading. We have

$$\mathbf{q} = \mathbf{k}\mathbf{d} + \mathbf{q}_0 \tag{15–5}$$

If the results are negative, it indicates that the loading acts in the opposite direction to that shown in Fig. 15–4.

PROCEDURE FOR ANALYSIS

The following method provides a means of determining the displacements, support reactions, and internal loadings for the members or finite elements of a statically determinate or statically indeterminate beam.

Notation

- Divide the beam into finite elements and arbitrarily identify each element and its nodes. Use a number written in a circle for a node and a number written in a square for a member. Usually an element extends between points of support, points of concentrated loads, and joints, or to points where internal loadings or displacements are to be determined.
- Specify the near and far ends of each element symbolically by directing an arrow along the element, with the head directed toward the far end.
- At each nodal point specify numerically the y and z code numbers. In all cases use the *lowest code numbers* to identify all the unconstrained degrees of freedom, followed by the remaining or highest numbers to identify the degrees of freedom that are constrained.
- From the problem, establish the known displacements $\mathbf{D}_k$ and known external loads $\mathbf{Q}_k$. Include any *reversed* fixed-end loadings if an element supports an intermediate load.

Structure Stiffness Matrix

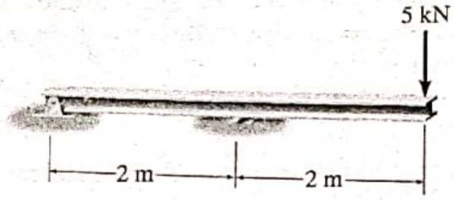
- Apply Eq. 15–1 to determine the stiffness matrix for each element expressed in global coordinates.
- After each member stiffness matrix is determined, and the rows and columns are identified with the appropriate code numbers, assemble the matrices to determine the structure stiffness matrix **K**. As a partial check, the member *and* structure stiffness matrices should all be *symmetric.*

Displacements and Loads

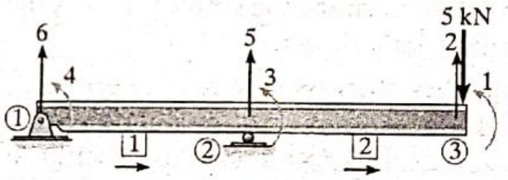
- Partition the structure stiffness equation and carry out the matrix multiplication in order to determine the unknown displacements $\mathbf{D}_u$ and support reactions $\mathbf{Q}_u$.
- The internal shear and moment $\mathbf{q}$ at the ends of each beam element can be determined from Eq. 15–5, accounting for the additional fixed-end loadings.

EXAMPLE 15-1

Determine the reactions at the supports of the beam shown in Fig. 15–8*a*. EI is constant.



(a)



(b)

Fig. 15–8

Solution

Notation. The beam has two elements and three nodes, which are identified in Fig. 15–8*b*. The code numbers 1 through 6 are indicated such that the *lowest numbers* 1–4 *identify the unconstrained degrees of freedom.*

The known load and displacement matrices are

$$\mathbf{Q}_k = \begin{array}{c} \begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \mathbf{D}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{array}{c} 5 \\ 6 \end{array}$$

Member Stiffness Matrices. Each of the two member stiffness matrices is determined from Eq. 15–1. Note carefully how the code numbers for each column and row are established.

$$\mathbf{k}_1 = EI \begin{array}{c} \begin{array}{cccc} 6 & 4 & 5 & 3 \end{array} \\ \begin{bmatrix} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ 6 \\ 4 \\ 5 \\ 3 \end{array} \qquad \mathbf{k}_2 = EI \begin{array}{c} \begin{array}{cccc} 5 & 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 3 \\ 2 \\ 1 \end{array}$$

Displacements and Loads. We can now assemble these elements into the structure stiffness matrix. For example, element $\mathbf{K}_{11} = 0 + 2 = 2$, $\mathbf{K}_{55} = 1.5 + 1.5 = 3$, etc. Thus,

$$\mathbf{Q} = \mathbf{KD}$$

$$\begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \\ \hdashline Q_5 \\ Q_6 \end{bmatrix} = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc:cc} 2 & -1.5 & 1 & 0 & 1.5 & 0 \\ -1.5 & 1.5 & -1.5 & 0 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 2 & -1.5 & 1.5 \\ \hdashline 1.5 & -1.5 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 1.5 & 1.5 & -1.5 & 1.5 \end{array}\right] \end{array} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ \hdashline 0 \\ 0 \end{bmatrix}$$

The matrices are partitioned as shown. Carrying out the multiplication for the first four rows, we have

$$0 = 2D_1 - 1.5D_2 + D_3 + 0$$

$$-\frac{5}{EI} = -1.5D_1 + 1.5D_2 - 1.5D_3 + 0$$

$$0 = D_1 - 1.5D_2 + 4D_3 + D_4$$

$$0 = 0 + 0 + D_3 + 2D_4$$

Solving,

$$D_1 = -\frac{16.67}{EI}$$

$$D_2 = -\frac{26.67}{EI}$$

$$D_3 = -\frac{6.67}{EI}$$

$$D_4 = \frac{3.33}{EI}$$

Using these results, and multiplying the last two rows, gives

$$Q_5 = 1.5EI\left(-\frac{16.67}{EI}\right) - 1.5EI\left(-\frac{26.67}{EI}\right) + 0 - 1.5EI\left(\frac{3.33}{EI}\right)$$

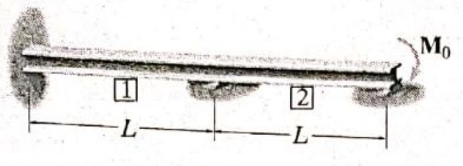
$$= 10 \text{ kN} \qquad \textit{Ans.}$$

$$Q_6 = 0 + 0 + 1.5EI\left(-\frac{6.67}{EI}\right) + 1.5EI\left(\frac{3.33}{EI}\right)$$

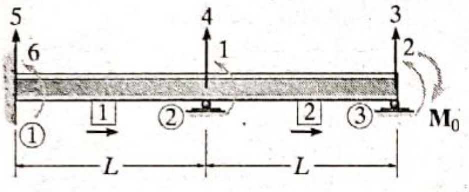
$$= -5 \text{ kN} \qquad \textit{Ans.}$$

EXAMPLE 15-2

Determine the internal shear and moment in member 1 of the beam shown in Fig. 15–9*a*. *EI* is constant.



(a)



(b)

Fig. 15–9

Solution

Notation. In this case the beam has only two unknown degrees of freedom, labeled with code numbers 1 and 2, Fig. 15–9*b*. Notice that the loading $\mathbf{M}_0$ is a negative quantity. The known load and displacement matrices are

$$\mathbf{Q}_k = \begin{array}{c} \begin{bmatrix} 0 \\ -M_0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \qquad \mathbf{D}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Member Stiffness Matrices. Applying Eq. 15–1 to each member, in accordance with the code numbers shown in Fig. 15–9*b*, we have

$$\mathbf{k}_1 = EI \begin{array}{c} \begin{array}{cccc} 5 & 6 & 4 & 1 \end{array} \\ \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 4 \\ 1 \end{array} \qquad \mathbf{k}_2 = EI \begin{array}{c} \begin{array}{cccc} 4 & 1 & 3 & 2 \end{array} \\ \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 1 \\ 3 \\ 2 \end{array}$$

Displacements and Loads. The structure stiffness matrix is formed by assembling the elements of the member stiffness matrices. Applying the structure matrix equation, we have

$$\mathbf{Q} = \mathbf{KD}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 \\ -M_0 \\ \hline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cc|cccc} \frac{8}{L} & \frac{2}{L} & -\frac{6}{L^2} & 0 & \frac{6}{L^2} & \frac{2}{L} \\ \frac{2}{L} & \frac{4}{L} & -\frac{6}{L^2} & \frac{6}{L^2} & 0 & 0 \\ \hline -\frac{6}{L^2} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{12}{L^3} & 0 & 0 \\ 0 & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{24}{L^3} & -\frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & 0 & 0 & -\frac{12}{L^3} & \frac{12}{L^3} & \frac{6}{L^2} \\ \frac{2}{L} & 0 & 0 & -\frac{6}{L^2} & \frac{6}{L^2} & \frac{4}{L} \end{array}\right] \end{array} \begin{bmatrix} D_1 \\ D_2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Multiplying the first two rows to determine the displacements, yields

$$0 = \frac{8EI}{L}D_1 + \frac{2EI}{L}D_2$$

$$-M_0 = \frac{2EI}{L}D_1 + \frac{4EI}{L}D_2$$

So that,

$$D_1 = \frac{M_0 L}{14EI}$$

$$D_2 = -\frac{2M_0 L}{7EI}$$

As in the last example, the reactions are obtained from the multiplication of the remaining rows. For example, the force reaction at the right support is

$$Q_3 = -\frac{6EI}{L^2}\left(\frac{M_0 L}{14EI}\right) - \frac{6EI}{L^2}\left(-\frac{2M_0 L}{7EI}\right) = \frac{9M_0}{7L}$$

The internal loadings at nodes 1 and 2 are determined from Eq. 15–2. We have

$$\mathbf{q} = \mathbf{k}_1\mathbf{d}$$

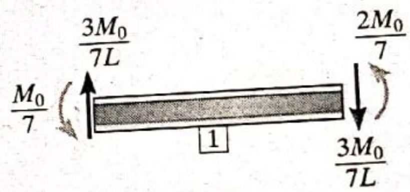
$$\begin{bmatrix} q_5 \\ q_6 \\ q_4 \\ q_1 \end{bmatrix} = EI \begin{array}{c} \begin{array}{cccc} 5 & 6 & 4 & 1 \end{array} \\ \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \end{array} \begin{array}{cc} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{M_0 L}{14EI} \end{bmatrix} & \begin{array}{c} 5 \\ 6 \\ 4 \\ 1 \end{array} \end{array}$$

$$q_5 = \frac{6EI}{L^2}\left(\frac{M_0 L}{14EI}\right) = \frac{3M_0}{7L}$$ *Ans.*

$$q_6 = \frac{2EI}{L}\left(\frac{M_0 L}{14EI}\right) = \frac{M_0}{7}$$ *Ans.*

$$q_4 = -\frac{6EI}{L^2}\left(\frac{M_0 L}{14EI}\right) = -\frac{3M_0}{7L}$$ *Ans.*

$$q_1 = \frac{4EI}{L}\left(\frac{M_0 L}{14EI}\right) = \frac{2M_0}{7}$$ *Ans.*



(c)

These results are shown in Fig. 15–9c.