CHAPTER 11

Displacement Method of Analysis: Slope-Deflection Equations

In this chapter we will briefly outline the basic ideas for analyzing structures using the displacement method of analysis. Once these concepts have been presented, we will develop the general equations of slope deflection and then use them to analyze statically indeterminate beams and frames.

11–1 Displacement Method of Analysis: General Procedures

All structures must satisfy equilibrium, load-displacement, and compatibility of displacements requirements in order to ensure their safety. It was stated in Sec. 10–1 that there are two different ways to satisfy these requirements when analyzing a statically indeterminate structure. The force method of analysis, discussed in the previous chapter, is based on identifying the unknown redundant forces and then satisfying the structure's compatibility equations. This is done by expressing the displacements in terms of the loads by using the load-displacement relations. The solution of the resultant equations yields the redundant reactions, and then the equilibrium equations are used to determine the remaining reactions on the structure.

The *displacement method* works the opposite way. It first requires satisfying equilibrium equations for the structure. To do this the unknown displacements are written in terms of the loads by using the load-displacement relations, then these equations are solved for the displacements. Once the displacements are obtained, the unknown loads are determined from the compatibility equations using the load-displacement relations. Every displacement method follows this general procedure. In this chapter, the procedure will be generalized to produce the slope-deflection equations. In Chapter 12, the moment-distribution method will be developed. This method sidesteps the calculation of the displacements and instead makes it possible to apply a series of converging corrections that allow direct calculation of the end moments. Finally, in Chapters 14, 15, and 16, we will illustrate how to apply this method using matrix analysis, making it suitable for use on a computer.

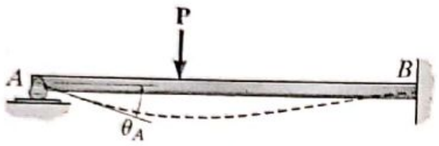
In the discussion that follows we will show how to identify the unknown displacements in a structure and we will develop some of the important load-displacement relations for beam and frame members. The results will be used in the next section and in later chapters as the basis for applying the displacement method of analysis.

Degrees of Freedom. When a structure is loaded, specified points on it, called *nodes*, will undergo unknown displacements. These displacements are referred to as the *degrees of freedom* for the structure, and in the displacement method of analysis it is important to specify these degrees of freedom since they become the unknowns when the method is applied. The number of these unknowns is referred to as the degree in which the structure is kinematically indeterminate.

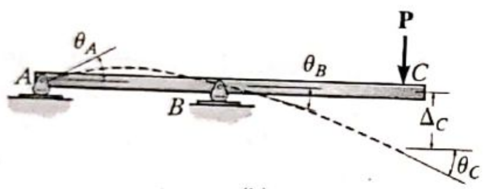
To determine the kinematic indeterminacy we can imagine the structure to consist of a series of members connected to nodes, which are usually located at *joints, supports,* at the *ends* of a member, or where the members have a sudden *change in cross section.* In three dimensions, each node on a frame or beam can have at most three linear displacements and three rotational displacements; and in two dimensions, each node can have at most two linear displacements and one rotational displacement. Furthermore, nodal displacements may be restricted by the supports, or due to assumptions based on the behavior of the structure. For example, if the structure is a beam and only deformation due to bending is considered, then there can be no linear displacement along the axis of the beam since this displacement is caused by axial-force deformation.

To clarify these concepts we will consider some examples, beginning with the beam in Fig. 11–1*a*. Here any load **P** applied to the beam will cause node *A* only to rotate (neglecting axial deformation), while node *B* is completely restricted from moving. Hence the beam has only one unknown degree of freedom, θ_A, and is therefore kinematically indeterminate to the first degree. The beam in Fig. 11–1*b* has nodes at *A*, *B*, and *C*, and so has four degrees of freedom, designated by the rotational displacements θ_A, θ_B, θ_C, and the vertical displacement Δ_C. It is kinematically indeterminate to the fourth degree. Consider now the frame in Fig. 11–1*c*. Again, if we neglect axial deformation of the members, an arbitrary loading **P** applied to the frame can cause nodes *B* and *C* to rotate, and these nodes can be displaced horizontally by an *equal* amount. The frame therefore has three degrees of freedom, θ_B, θ_C, Δ_B, and thus it is kinematically indeterminate to the third degree.

In summary, specifying the kinematic indeterminacy or the number of unconstrained degrees of freedom for the structure is a necessary first



(a)



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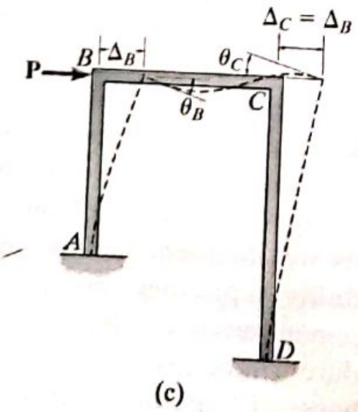


(c)

Fig. 11–1

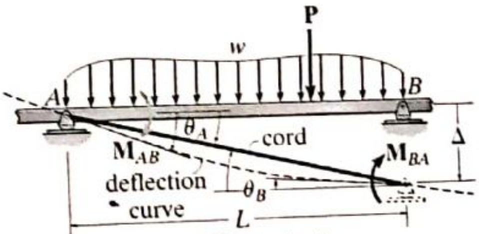
step when applying a displacement method of analysis. It identifies the number of unknowns in the problem, based on the assumptions made regarding the deformation behavior of the structure. Furthermore, once these nodal displacements are known, the deformation of the structural members can be completely specified, and the loadings within the members obtained.

11–2 Slope-Deflection Equations

As indicated previously, the method of consistent displacements studied in Chapter 10 is called a force method of analysis, because it requires writing equations that relate the unknown forces or moments in a structure. Unfortunately, its use is limited to structures which are *not* highly indeterminate. This is because much work is required to set up the compatibility equations, and furthermore each equation written involves *all the unknowns*, making it difficult to solve the resulting set of equations unless a computer is available. By comparison, the slope-deflection method is not as involved. As we shall see, it requires less work both to write the necessary equations for the solution of a problem and to solve these equations for the unknown displacements and associated internal loads. Also, the method can be easily programmed on a computer and used to analyze a wide range of indeterminate structures.

The slope-deflection method was originally developed by Heinrich Manderla and Otto Mohr for the purpose of studying secondary stresses in trusses. Later, in 1915, G. A. Maney developed a refined version of this technique and applied it to the analysis of indeterminate beams and framed structures.

General Case. The slope-deflection method is so named since it relates the unknown slopes and deflections to the applied load on a structure. In order to develop the general form of the slope-deflection equations, we will consider the typical span AB of a continuous beam as shown in Fig. 11–2, which is subjected to the arbitrary loading and has a constant EI. We wish to relate the beam's internal end moments M_{AB} and M_{BA} in terms of its three degrees of freedom, namely, its angular displacements θ_A and θ_B, and linear displacement Δ which could be caused by a relative settlement between the supports. Since we will be developing a formula, *moments* and *angular displacements* will be considered *positive* when they act *clockwise on the span*, as shown in Fig. 11–2. Furthermore, the *linear displacement* Δ is considered *positive* as shown, since this displacement causes the cord of the span and the span's cord angle ψ to rotate *clockwise*.



EI is constant
positive sign convention

Fig. 11–2

The slope-deflection equations can be obtained by using the principle of superposition by considering *separately* the moments developed at each support due to each of the displacements, θ_A, θ_B, and Δ, and then the loads.

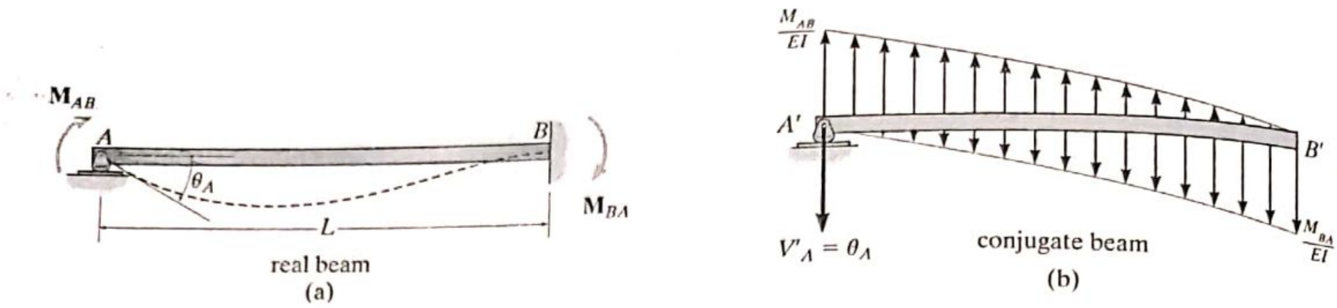


Fig. 11–3

Angular Displacement at A, θ_A. Consider node A of the member shown in Fig. 11–3a to rotate θ_A while its far-end node B is *held fixed*. To determine the moment M_{AB} needed to cause this displacement, we will use the conjugate-beam method. For this case the conjugate beam is shown in Fig. 11–3b. Notice that the end shear at A' acts downward on the beam, since θ_A is clockwise. The deflection of the "real beam" in Fig. 11–3a is to be zero at A and B, and therefore the corresponding sum of the *moments* at each end A' and B' of the conjugate beam must also be zero. This yields

$$\circlearrowleft + \Sigma M_{A'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

from which we obtain the following load-displacement relationships.

$$\boxed{M_{AB} = \frac{4EI}{L}\theta_A} \tag{11–1}$$

$$\boxed{M_{BA} = \frac{2EI}{L}\theta_A} \tag{11–2}$$

Angular Displacement at B, θ_B. In a similar manner, if end B of the beam rotates to its final position θ_B, while end A is *held fixed*, Fig. 11–4, we can relate the applied moment M_{BA} to the angular displacement θ_B and the reaction moment M_{AB} at the wall. The results are

$$\boxed{M_{BA} = \frac{4EI}{L}\theta_B} \tag{11–3}$$

$$\boxed{M_{AB} = \frac{2EI}{L}\theta_B} \tag{11–4}$$

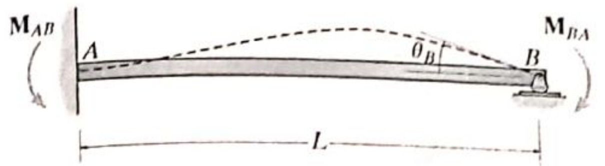


Fig. 11–4

Relative Linear Displacement, Δ. If the far node B of the member is displaced relative to A, so that the cord of the member rotates clockwise (positive displacement) and yet both ends do not rotate, then equal but opposite moment and shear reactions are developed in the member, Fig. 11–5*a*. As before, the moment **M** can be related to the displacement Δ using the conjugate-beam method. In this case, the conjugate beam, Fig. 11–5*b*, is free at both ends, since the real beam (member) is fixed supported. However, due to the *displacement* of the real beam at B, the *moment* at the end B' of the conjugate beam must have a magnitude of Δ as indicated.* Summing moments about B', we have

$$\circlearrowright+\Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{2}{3}L\right)\right] - \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{1}{3}L\right)\right] - \Delta = 0$$

$$\boxed{M_{AB} = M_{BA} = M = \frac{-6EI}{L^2}\Delta} \qquad (11\text{–}5)$$

By our sign convention, this induced moment is negative since for equilibrium it acts counterclockwise on the member.

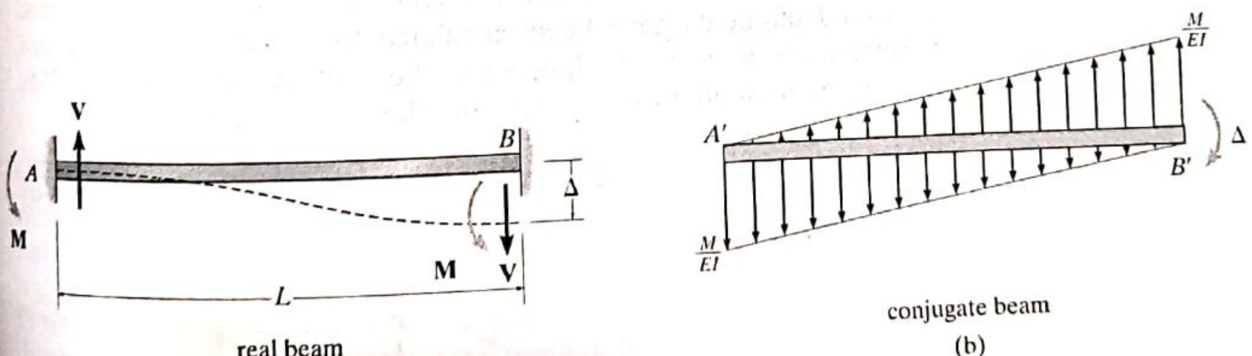


Fig. 11–5

*The moment diagrams shown on the conjugate beam were determined by the method of superposition for a simply supported beam, as explained in Sec. 4–5.

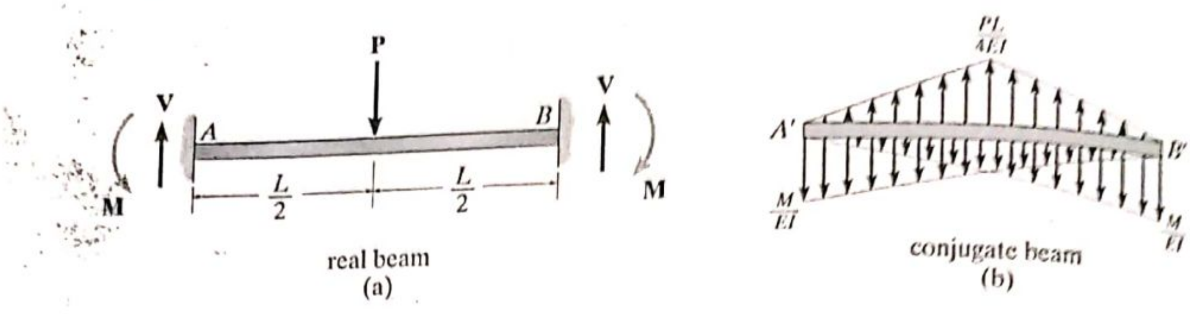


Fig. 11–6

Fixed-End Moments. In the previous cases we have considered relationships between the displacements and the necessary moments M_{AB} and M_{BA} acting at nodes A and B, respectively. In general, however, the linear or angular displacements of the nodes are caused by loadings acting on the *span* of the member, not by moments acting at its nodes. In order to develop the slope-deflection equations, we must transform these *span loadings* into equivalent moments acting at the nodes and then use the load-displacement relationships just derived. This is done simply by finding the reaction moment that each load develops at the nodes. For example, consider the fixed-supported member shown in Fig. 11–6*a*, which is subjected to a concentrated load **P** at its center. The conjugate beam for this case is shown in Fig. 11–6*b*. Since we require the slope at each end to be zero,

$$+\uparrow \Sigma F_y = 0; \qquad \left[\frac{1}{2}\left(\frac{PL}{4EI}\right)L\right] - 2\left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right] = 0$$

$$M = \frac{PL}{8}$$

This moment is called a *fixed-end moment* (FEM). Note that according to our sign convention, it is negative at node A (counterclockwise) and positive at node B (clockwise). For convenience in solving problems, fixed-end moments have been calculated for other loadings and are tabulated on the inside back cover of the book. Assuming these FEMs have been computed for a specific problem (Fig. 11–7), we have

$$M_{AB} = (\text{FEM})_{AB} \qquad M_{BA} = (\text{FEM})_{BA} \tag{11–6}$$

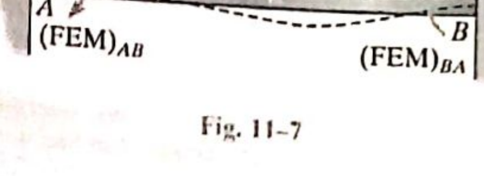


Fig. 11–7

Slope-Deflection Equation. If the end moments due to each displacement (Eqs. 11–1 through 11–5) and the loading (Eq. 11–6) are added together, the resultant moments at the ends can be written as

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{BA} \qquad (11\text{–}7)$$

Since these two equations are similar, the result can be expressed as a single equation. Referring to one end of the span as the near end (N) and the other end as the far end (F), and letting the *member stiffness* be represented as $k = I/L$, and the *span's cord rotation* as ψ (psi) $= \Delta/L$, we can write

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N \qquad (11\text{–}8)$$

For Internal Span or End Span with Far End Fixed

where

M_N = internal moment in the near end of the span; this moment is *positive clockwise* when acting on the span.

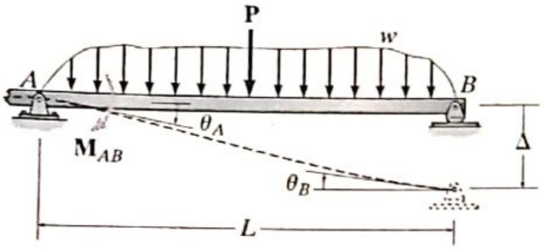
E, k = modulus of elasticity of material and span stiffness $k = I/L$.

θ_N, θ_F = near- and far-end slopes or angular displacements of the span at the supports; the angles are measured in *radians* and are *positive clockwise.*

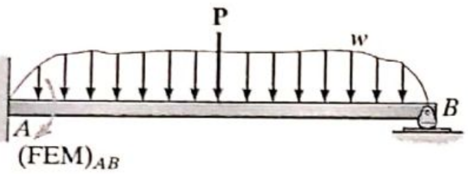
ψ = span rotation of its cord due to a linear displacement, that is, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise.*

$(\text{FEM})_N$ = fixed-end moment at the near-end support; the moment is *positive clockwise* when acting on the span; refer to the table on the inside back cover for various loading conditions.

From the derivation Eq. 11–8 is both a compatibility and load-displacement relationship found by considering only the effects of bending and neglecting axial and shear deformations. It is referred to as the general *slope-deflection equation.* When used for the solution of problems, this equation is applied *twice* for each member span (AB); that is, application is from A to B and from B to A for span AB in Fig. 11–2.



(a)



(b)

Fig. 11–8

Pin-Supported End Span. Occasionally an end span of a beam or frame is supported by a pin or roller at its *far end,* Fig. 11–8*a*. When this occurs, the moment at the roller or pin must be zero; and provided the angular displacement θ_B at this support does not have to be determined, we can modify the general slope-deflection equation so that it has to be applied *only once* to the span rather than twice. To do this we will apply Eq. 11–8 or Eqs. 11–7 to each end of the beam in Fig. 11–8. This results in the following two equations:

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$
$$0 = 2Ek(2\theta_F + \theta_N - 3\psi) + 0 \qquad (11\text{–}9)$$

Here the $(\text{FEM})_F$ is equal to zero since the far end is pinned, Fig. 11–8*b*. Furthermore, the $(\text{FEM})_N$ can be obtained, for example, using the table in the right-hand column on the inside back cover of this book. Multiplying the first equation by 2 and subtracting the second equation from it *eliminates* the unknown θ_F and yields

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N \qquad (11\text{–}10)$$

Only for End Span with Far End Pinned or Roller Supported

Since the moment at the far end is zero, only *one* application of this equation is necessary for the end span. This simplifies the analysis since the general equation, Eq. 11–8, would require *two* applications for this span and therefore involve the (extra) unknown angular displacement θ_B (or θ_F) at the end support.

To summarize application of the slope-deflection equations, consider the continuous beam shown in Fig. 11–9 which has four degrees of freedom. Here Eq. 11–8 can be applied twice to each of the three spans, i.e., from *A* to *B*, *B* to *A*, *B* to *C*, *C* to *B*, *C* to *D*, and *D* to *C*. These equations would involve the four unknown rotations, θ_A, θ_B, θ_C, θ_D. Since the end moments at *A* and *D* are zero, however, it is not necessary to determine θ_A and θ_D. A shorter solution occurs if we apply Eq. 11–10 from *B* to *A* and *C* to *D* and then apply Eq. 11–8 from *B* to *C* and *C* to *B*. These four equations will involve only the unknown rotations θ_B and θ_C.

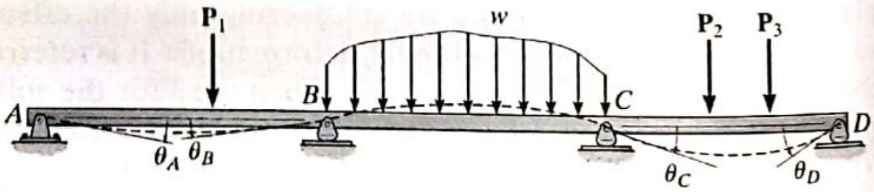


Fig. 11–9

11-3 Analysis of Beams

PROCEDURE FOR ANALYSIS

Degrees of Freedom. Label all the supports and joints (nodes) in order to identify the spans of the beam or frame between the nodes. By drawing the deflected shape of the structure, it will be possible to identify the number of degrees of freedom. Here each node can possibly have an angular displacement and a linear displacement. *Compatibility* at the nodes is maintained provided the members that are fixed connected to a node undergo the same displacements as the node. If these displacements are unknown, and in general they will be, then for convenience *assume* they act in the *positive direction* so as to cause *clockwise* rotation of a member or joint, Fig. 11–2.

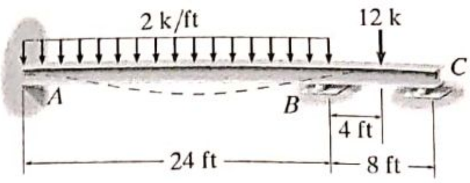
Slope-Deflection Equations. The slope-deflection equations relate the unknown moments applied to the nodes to the displacements of the nodes for any span of the structure. If a load exists on the span, compute the FEMs using the table given on the inside back cover. Also, if a node has a linear displacement, Δ, compute $\psi = \Delta/L$ for the adjacent spans. Apply Eq. 11–8 to each end of the span, thereby generating *two* slope-deflection equations for each span. However, if a span at the *end* of a continuous beam or frame is pin supported, apply Eq. 11–10 only to the restrained end, thereby generating *one* slope-deflection equation for the span.

Equilibrium Equations. Write an equilibrium equation for each unknown degree of freedom for the structure. Each of these equations should be expressed in terms of unknown internal moments as specified by the slope-deflection equations. For beams and frames write the moment equation of equilibrium at each support, and for frames also write joint moment equations of equilibrium. If the frame sidesways or deflects horizontally, column shears should be related to the moments at the ends of the column. This is discussed in Sec. 11–5.

Substitute the slope-deflection equations into the equilibrium equations and solve for the unknown joint displacements. These results are then substituted into the slope-deflection equations to determine the internal moments at the ends of each member. If any of the results are *negative,* they indicate *counterclockwise* rotation; whereas *positive* moments and displacements are applied *clockwise.*

EXAMPLE 11-2

Draw the shear and moment diagrams for the beam shown in Fig. 11–11*a*. EI is constant.



(a)

Fig. 11-11

Solution

Slope-Deflection Equations. Two spans must be considered in this problem. Equation 11–8 applies to span AB. We can use Eq. 11–10 for span BC since the *end C is on a roller*. Using the formulas for the FEMs tabulated on the inside back cover, we have

$$(\text{FEM})_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(2)(24)^2 = -96\text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{BA} = \frac{wL^2}{12} = \frac{1}{12}(2)(24)^2 = 96\text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{BC} = -\frac{3PL}{16} = -\frac{3(12)(8)}{16} = -18\text{ k}\cdot\text{ft}$$

Note that $(\text{FEM})_{AB}$ and $(\text{FEM})_{BC}$ are negative since they act counterclockwise on the beam at A and B, respectively. Also, since the supports do not settle, $\psi_{AB} = \psi_{BC} = 0$. Applying Eq. 11–8 for span AB and realizing that $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{24}\right)[2(0) + \theta_B - 3(0)] - 96$$

$$M_{AB} = 0.08333EI\theta_B - 96 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{24}\right)[2\theta_B + 0 - 3(0)] + 96$$

$$M_{BA} = 0.1667EI\theta_B + 96 \qquad (2)$$

Applying Eq. 11–10 with B as the near end and C as the far end, we have

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

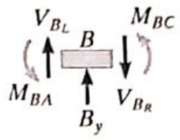
$$M_{BC} = 3E\left(\frac{I}{8}\right)(\theta_B - 0) - 18$$

$$M_{BC} = 0.375EI\theta_B - 18 \qquad (3)$$

Remember that Eq. 11–10 is *not* applied from C (near end) to B (far end).

Equilibrium Equations. The above three equations contain four unknowns. The necessary fourth equation comes from the conditions of equilibrium at the support B. The free-body diagram is shown in Fig. 11–11b. We have

$$\circlearrowleft + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \tag{4}$$



(b)

To solve, substitute Eqs. (2) and (3) into Eq. (4), which yields

$$\theta_B = -\frac{144.0}{EI}$$

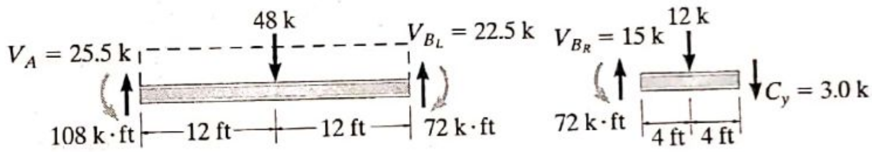
Since θ_B is negative (counterclockwise) the elastic curve for the beam has been correctly drawn in Fig. 11–11a. Substituting θ_B into Eqs. (1)–(3), we get

$$M_{AB} = -108.0 \text{ k} \cdot \text{ft}$$

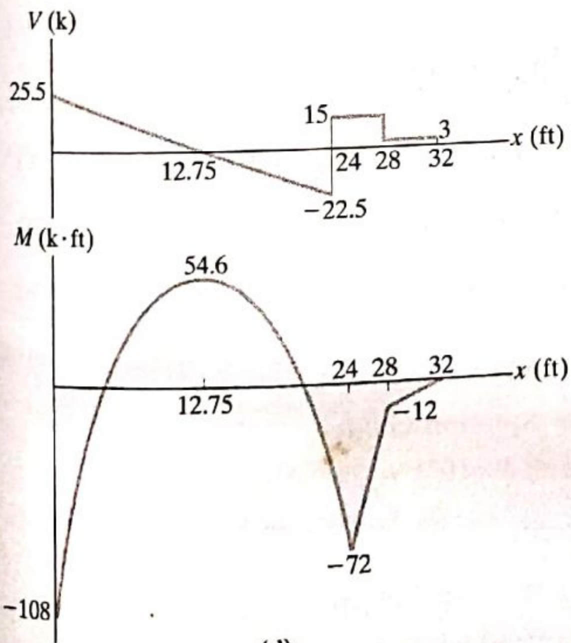
$$M_{BA} = 72.0 \text{ k} \cdot \text{ft}$$

$$M_{BC} = -72.0 \text{ k} \cdot \text{ft}$$

Using these data for the moments, the shear reactions at the ends of the beam spans have been determined in Fig. 11–11c. The shear and moment diagrams are plotted in Fig. 11–11d.



(c)



(d)