Sampling Theorem

Sampling of the signals is the fundamental operation in signal-processing. A continuous time signal is first converted to discrete-time signal by sampling process. The sufficient number of samples of the signal must be taken so that the original signal is represented in its samples completely. Also, it should be

possible to recover or reconstruct the original signal completely from its samples. The number of samples to be taken depends on maximum signal frequency present in the signal.
Sampling theorem gives the complete idea about the sampling of signals. Different types of samples are ideal samples, natural samples and flat-top sampling.

The Sampling theorem may be stated as
" A Continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is $f_s \geq 2 f_m$.
Here, f_s is the sampling frequency and f_m is the maximum frequency present in the signal.

Proof of Sampling Theorem

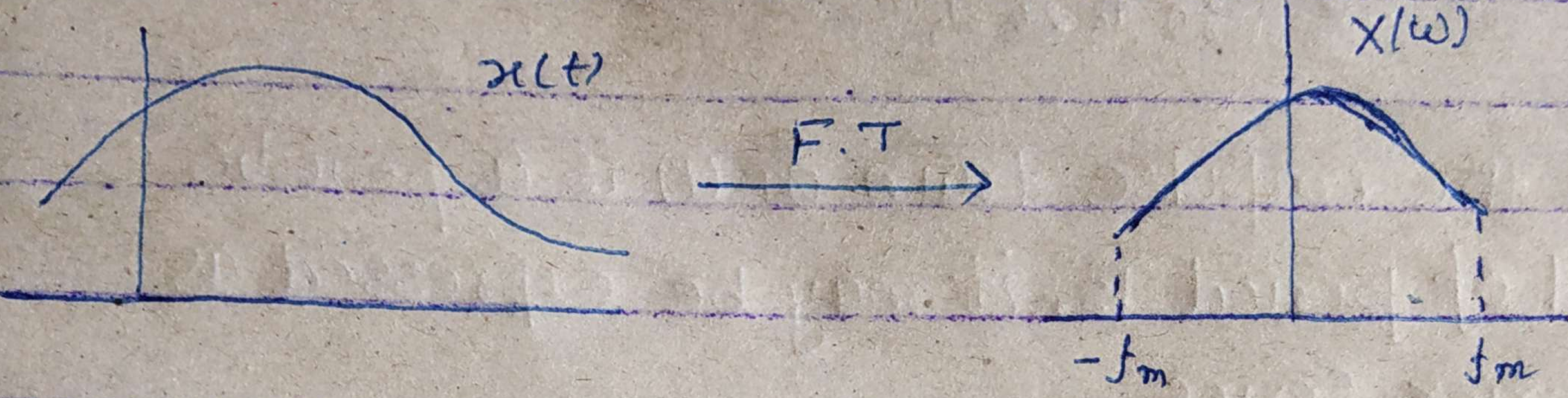
Let us consider a continuous time signal $x(t)$ whose spectrum is bandlimited to f_m Hz. This means that the signal $x(t)$ has no frequency components beyond f_m Hz. Therefore, $X(\omega)$ is zero for $|\omega| > \omega_m$ i.e

$X(\omega) = 0$ for $|\omega| > \omega_m$

where $\omega_m = 2\pi f_m$

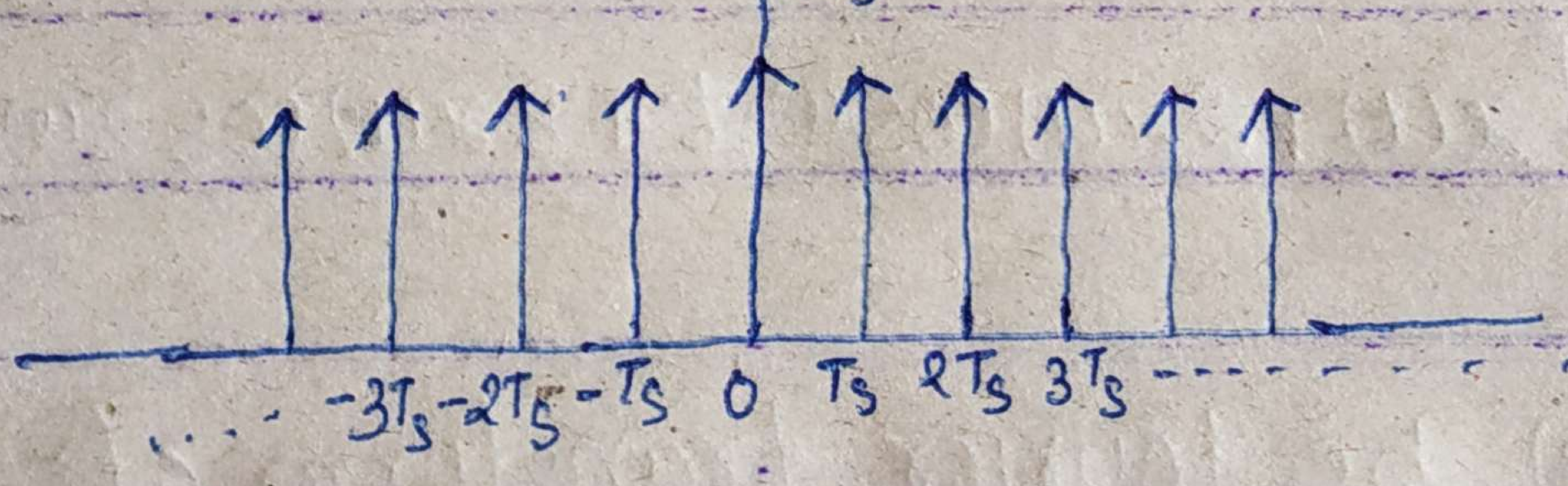
Let $x(t)$ be continuous - time signal whose fourier transform is given as $X(\omega)$

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

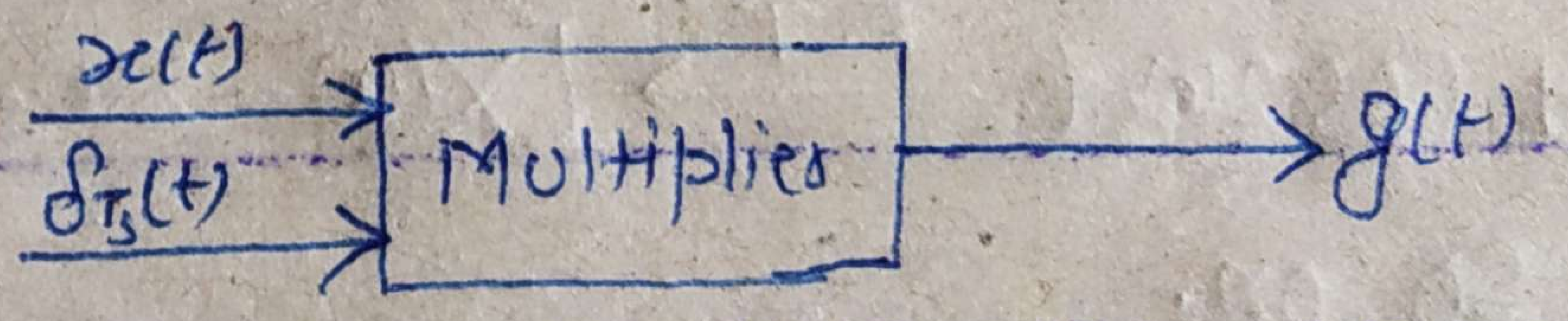


Sampling of $x(t)$ at a rate of f_s Hz may be achieved by multiplying $x(t)$ by an impulse train ~~$\delta_{Ts}(t)$~~ $\delta_{T_s}(t)$. The impulse train $\delta_{T_s}(t)$ Consists of unit impulses repeating periodically every T_s seconds, where $T_s = \frac{1}{f_s}$.

$f_s >> f_m$.

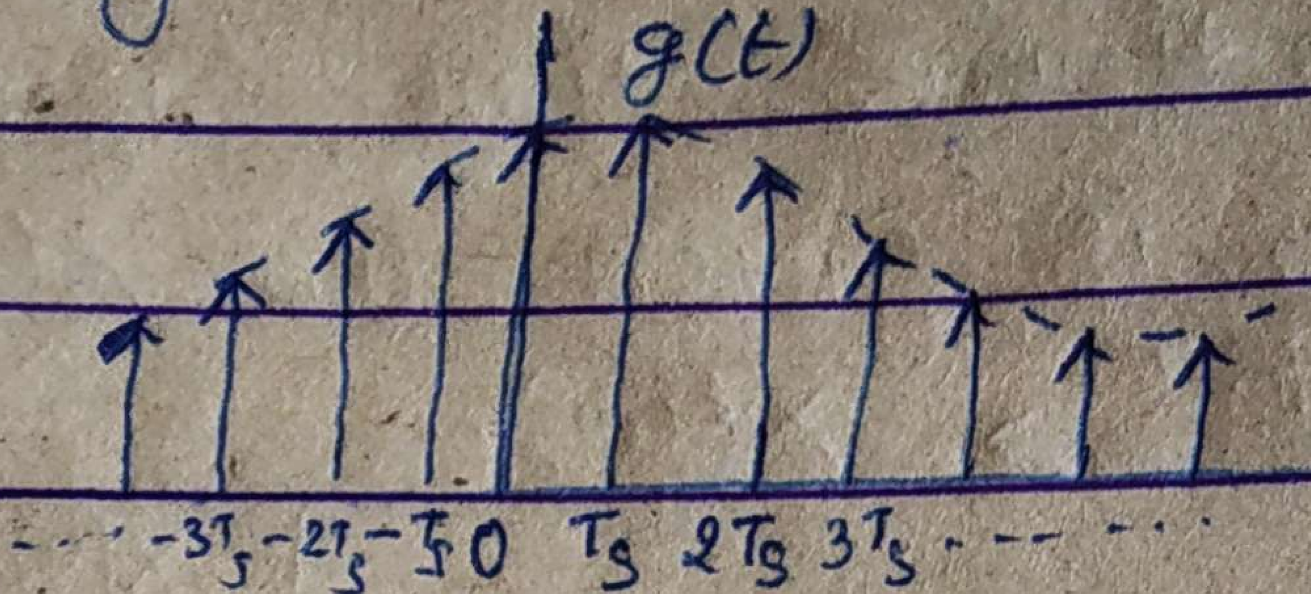


This mutiplication results in the sampled Signat $g(t)$



This sampled Signal consists of impulses spaced every T_s Seconds (the sampling interval). The resulting or sampled signal may be written as

$$g(t) = x(t)\,\delta_{T_s}(t)$$



Since the impulse train $\delta_{T_s}(t)$ is a periodic signal of period T_s, it may be expressed as a Fourier Series.

The trigonometric Fourier series expansion of impulse-train $\delta_{T_s}(t)$ is expressed as

$$\delta_{T_s}(t) = \frac{1}{T_s}\left[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \cdots\right]$$

where $\omega_s = \dfrac{2\pi}{T_s} = 2\pi f_s$

$$\therefore\; g(t) = \frac{1}{T_s}\left[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + 2x(t)\cos 3\omega_s t + \cdots\right]$$

Let $G(\omega)$ be fourier transform of $g(t)$

& $x(t) \xrightarrow{F.T} X(\omega)$

$$2x(t)\cos\omega_s t \xrightarrow{F.T} \left[X(\omega-\omega_s) + X(\omega+\omega_s)\right]$$

$$2x(t)\cos 2\omega_s t \xrightarrow{F.T} \left[X(\omega-2\omega_s) + X(\omega+2\omega_s)\right]$$

and so on.

Therefore

time domain equation

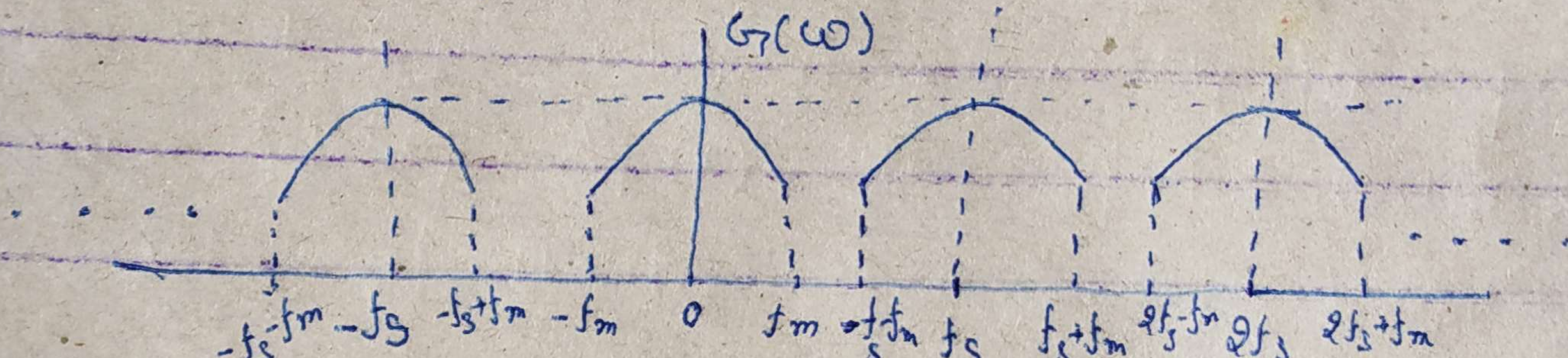
$$g(t) = x(t)\,\delta_{T_s}(t)$$

can be written in frequency domain as

$$G(\omega) = \frac{1}{T_s}\left[X(\omega) + X(\omega-\omega_s) + X(\omega+\omega_s) + X(\omega-2\omega_s) + X(\omega+2\omega_s) + X(\omega-3\omega_s) + X(\omega+3\omega_s) + \cdots\right]$$

$$G(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

it is clear from above equation that the spectrum $G(\omega)$ consists of $X(\omega)$ repeating periodically with period $\omega_s = 2\pi f_s$



to reconstruct $x(t)$ from $g(t)$, we must be able to recover $X(\omega)$ from $G(\omega)$, this is possible there is no overlap between successive cycles of $G(\omega)$. This requires

$$f_s \geq 2 f_m$$

Sampling interval $T_s = \dfrac{1}{f_s}$

Hence, $T_s < \dfrac{1}{2 f_m}$

Therefore, as long as the sampling frequency f_s is greater than twice than maximum signal frequency f_m (signal bandwidth f_m), $G(\omega)$

will consist of non-overlapping repetitions of $X(\omega)$. $x(t)$ can be recovered from its samples $g(t)$ by passing the sampled signal $g(t)$ through an ideal low-pass filter of bandwidth f_m Hz.