8

Slope and Deflection

8·1 Introduction

In certain situations it becomes necessary to design a machine component or a structure to minimise deflection. Thus it becomes essential to determine the deflection of the member. Also for statically indeterminate or fixed beams it becomes necessary to determine the slope and deflection at salient points. We shall deal with these problems in this chapter.

8·2 Bending with Uniform Curvature

Consider a beam *ACB* of length l, bent in the form of a circular arc *ADB*. Let *CD* be the deflection equal to y and *OB* be the radius of curvature equal to R as shown in Fig. 8·1.

Now $AC \times CB = FC \times CD$

$$\frac{l^2}{4} = (2R - y)\,y = 2Ry - y^2$$

$$y^2 - 2Ry + \frac{l^2}{4} = 0$$

$$y = \frac{2R \pm \sqrt{4R^2 - l^2}}{2} \qquad \text{...(8·1)}$$

Also $$\frac{1}{R} = \frac{M}{EI}$$

Hence y can be determined.

Let the tangent at B makes an angle θ with the axis of the beam.

Slope at $$B = \tan\theta = \frac{dy}{dx}$$

For θ to be small, $\tan\theta \cong \theta \cong \sin\theta$

Now $$\sin\theta = \frac{BC}{OB} = \frac{l}{2R}$$

$\therefore$ $$\theta = \frac{Ml}{2EI} \qquad \text{...(8·2)}$$

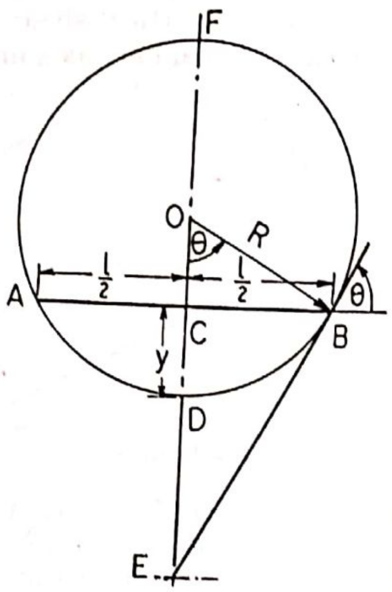

Fig. 8·1 Bending of a beam with uniform curvature.

8·3 Differential Equation of the Deflection Curve

Consider a beam *AB* which takes the curved shape as shown in Fig. 8·2 (*a*). Consider an elementary length *CD* equal to ds of the beam. Let the tangent to the elastic curve at C makes with the x-axis of the beam an angle θ. The angle at D will decrease and let it be $(\theta - d\theta)$. The normals at C and D intersect at O.

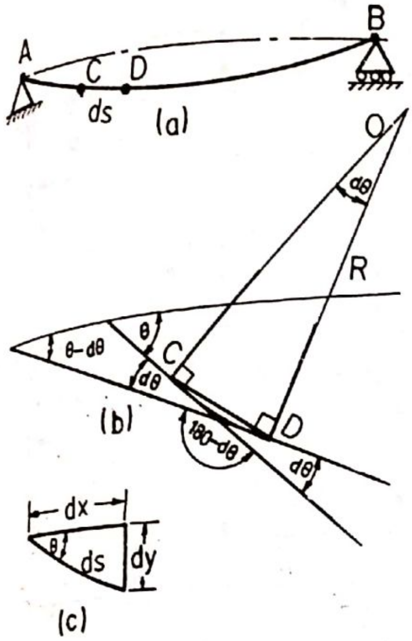

Fig. 8·2 Deriving differential equation of deflection curve.

If R is the radius of curvature of the bent beam, then [Fig. 8·2 (*b*)],

$$OC = OD = R$$

and $$\angle COD = d\theta$$

$$\therefore \quad CD = ds = R d\theta$$

$$\frac{1}{R} = \frac{d\theta}{ds}$$

Now from Fig. 8·2 (*c*),

$$\tan\theta = \frac{dy}{dx}$$

where y = Deflection of the beam

Differentiating with respect to s, we get

$$\frac{d}{ds}\tan\theta = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

$$\sec^2\theta \, . \frac{d\theta}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds}$$

Now $$\frac{dx}{ds} = \cos\theta$$

$$\therefore \quad \sec^2 . \frac{d\theta}{ds} = \frac{d^2y}{dx^2}$$

$$\frac{d\theta}{ds} = \frac{d^2y}{dx^2} \cdot \frac{1}{\sec^3\theta}$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\therefore \quad \frac{d\theta}{ds} = \frac{d^2y}{dx^2} \cdot \frac{1}{(1+\tan^2\theta)^{3/2}}$$

$$\frac{d\theta}{ds} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}} = \frac{1}{R} \qquad \text{...(8·3 a)}$$

This is the well known Newton's formula for curvature.

If the curvature is very small, then $\frac{dy}{dx}$ is also small and its square is negligible. Hence

$$\frac{1}{R} \approx \frac{d^2y}{dx^2}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \qquad \text{...(8·3b)}$$

The bending moment has been taken as positive if tension is caused in the bottom fibres of the beam. The positive bending moment should cause positive curvature. The curvature is taken positive if the centre of curvature O is above the bent curve of the beam as shown in Fig. 8·2 (*b*). If the curvature is positive, the angle θ decreases in going from C to D. Hence with proper sign

$$\frac{1}{R} = -\frac{d\theta}{ds} = -\frac{M}{EI} = -\frac{d^2y}{dx^2}$$

$\therefore$
$$-M = EI\frac{d^2y}{dx^2} \qquad \text{...(8·4)}$$

Slope of the beam,
$$\frac{dy}{dx} = -\int \frac{M}{EI}\,dx \qquad \text{...(8·5)}$$

Deflection,
$$y = -\iint \frac{M}{EI}\,dx.dx \qquad \text{...(8·6)}$$

Shear force,
$$F = \frac{dM}{dx} = -EI\frac{d^2y}{dx^3} \qquad \text{...(8·7)}$$

and uniform load,
$$w = -EI\frac{d^4y}{dx^4} \qquad \text{...(8·8)}$$

8·4 Methods of Solution

The following methods are generally used to determine the slope and deflection at a point of a beam.

(*i*) Macaulay's method. (*ii*) Moment area method. (*iii*) Conjugate beam method.

8·4·1 Macaulay's Method. This method is suitable for cases of beams subjected to concentrated loads and can be extended to uniformly distributed loads also. It consists of successive integration of expressions for bending moment in such a way that same constants of integration are valid for all portions of the beam even though the law of bending moment differs from portion to portion. For the beam shown in Fig. 8·3, the expressions for B.M. in different parts of the beam are :

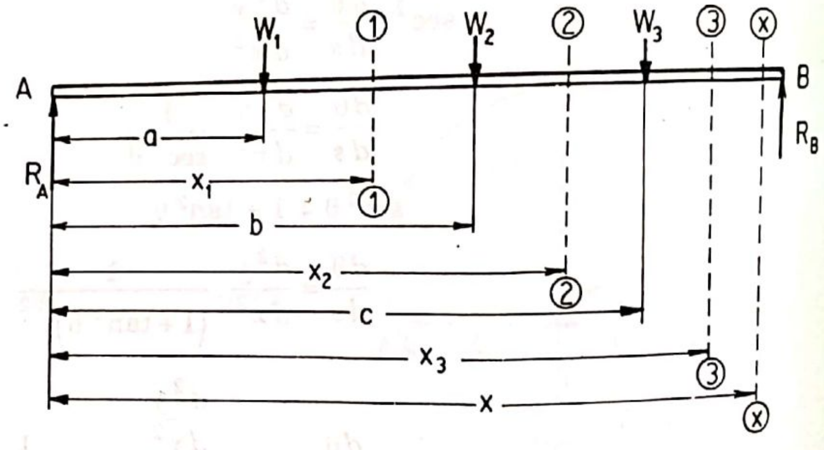

Fig. 8·3 Beam subjected to concentrated loads.

At section (1)–(1), $M_{x1} = R_A \times x_1 - W_1\,(x_1 - a)$

At section (2)–(2), $M_{x2} = R_A \times x_2 - W_1\,(x_2 - a) - W_2\,(x_2 - b)$

At section (3)–(3), $M_{x3} = R_A \times x_3 - W_1\,(x_3 - a) - W_2\,(x_3 - b) - W_2\,(x_3 - c)$

At section (x)–(x), $M_x = R_A \times x - W_1\,(x - a) - W_2\,(x - b) - W_3\,(x - c)$

is the general expression for bending moment. The general expression holds if the convention is adopted that a bracket is disregarded when for a particular value of x the contents of the bracket becomes negative. Therefore, it follows that bracket terms must be kept in tact and not multiplied out until a numerical substitution is made for x. The form of integration for a particular terms is

$$\int W(x-a)\,dx = \frac{W(x-a)^2}{2} \qquad \text{...(8·10)}$$

The general expression for bending moment must be written by taking a section just before the right hand support or load.

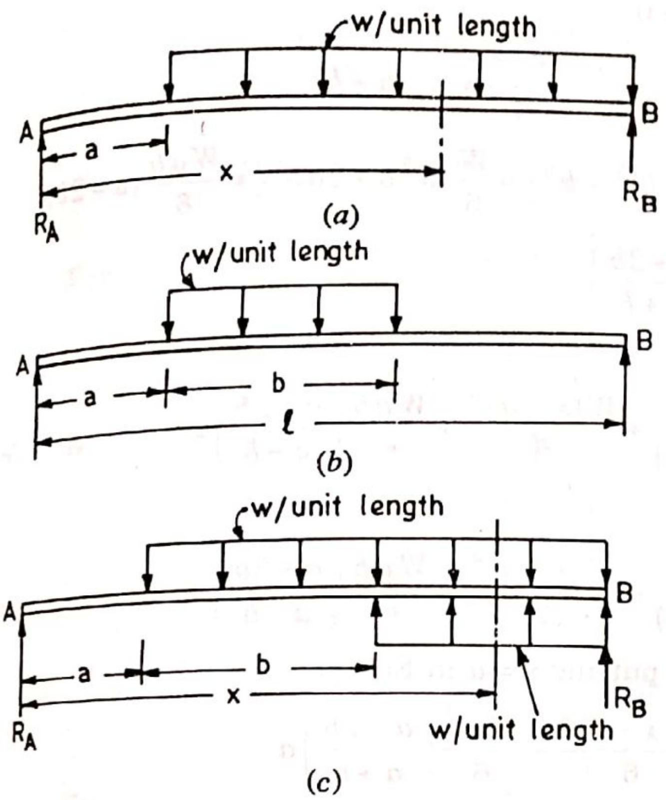

Fig. 8·4 Beam having u.d.l. on partial length.

If there is a uniformly distributed load starting anywhere and extending upto the right end then the general equation for bending moment will hold good for the entire beam. However, if the load does not extend upto the right end, the load may be extended upto the right end, and an equal and opposite load may be added to counteract the effect due to the additional load. For example, for the beam shown in Fig. 8·4 (*a*),

$$M_x = R_A \times x - \frac{w(x-a)^2}{2}$$

whereas for the beam shown in Fig. 8·4 (*b*), the load is not extending upto the right end. In Fig. 8·4 (*c*) the load is extended upto the right end and an equal and opposite load is applied on the portion of the beam to the right of the load which was earlier unloaded. In this case, the bending moment expression is

$$M_x = R_A \times x - \frac{w(x-a)^2}{2} + \frac{w(x-a-b)^2}{2}$$

8·4·1·1 Point load on a simply supported beam. Consider a beam of length l, simply supported at the ends and carrying a concentrated load W at a distance a from left end A. Taking a section at a distance x from left end A as shown in Fig. 8·5,

$$M_x = R_A \times x - W(x-a) \quad \text{...(a)}$$

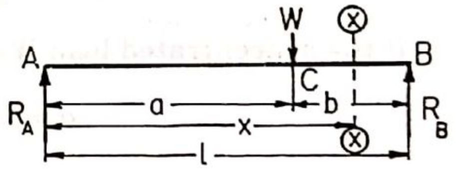

Fig. 8·5 Beam having a point load.

Now $\quad R_A = \dfrac{Wb}{a+b} = \dfrac{Wb}{l}$

$$R_B = \frac{Wa}{a+b} = \frac{Wa}{l}$$

Differential equation of bending becomes :

$$El = \frac{d^2y}{dx^2} = -M_x = -R_A \times x + W(x-a)$$

$$= -\frac{Wbx}{a+b} + W(x-a) \quad \text{...(b)}$$

Integrating, we get $\quad El\dfrac{dy}{dx} = -\dfrac{Wbx^2}{2(a+b)} + \dfrac{W(x+a)^2}{2} + C_1 \quad$ ...(*c*)

where C_1 is a constant of integration.

Integrating again, we get $\quad EI\,y = -\dfrac{Wbx^3}{6(a+b)} + \dfrac{W(x-a)^3}{6} + C_1x + C_2 \quad$ ...(*d*)

where C_2 is another constant of integration.

Now the end conditions are :

At $\quad x = 0,\; y = 0$

$\therefore \quad C_2 = 0$

Also at $x = a + b,\ y = 0$

$$\therefore \quad 0 = -\frac{Wb}{6}(a+b)^2 + \frac{Wb^2}{6} + C_1(a+b)$$

$$C_1(a+b) = \frac{W}{6}\left\{b(a+b)^2 - b^3\right\} = \frac{W}{6}\left[a^2b + 2ab^2\right] = \frac{Wab}{6}(a+2b)$$

$$\therefore \quad C_1 = \frac{Wab}{6}\left(\frac{a+2b}{a+b}\right)$$

Substituting in Eq. (b), we get

$$EIy = -\frac{Wbx^2}{6(a+b)} + \frac{W(x-a)^3}{6} + \frac{Wab}{6}\left(\frac{a+2b}{a+b}\right)x \qquad \text{...(e)}$$

Similarly, Eq. (c) becomes :

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2(a+b)} + \frac{W(x-a)^2}{2} + \frac{Wab}{6}\left(\frac{a+2b}{a+b}\right) \qquad \text{...(f)}$$

Deflection under the load is obtained by putting $x = a$ in Eq. (e).

$$EIy_c = -\frac{Wba^3}{6(a+b)} + \frac{W(x-a)^2}{6} + \frac{Wab}{6}\left(\frac{a+2b}{a+b}\right)a$$

$$= -\frac{Wba^3}{6(a+b)} + \frac{Wa^2b}{6}\left(\frac{a+2b}{a+b}\right) = \frac{Wa^2b}{6(a+b)}\ [a+2b-a] = \frac{Wa^2b^2}{3(a+b)}$$

$$\therefore \quad y_c = \frac{Wa^2b^2}{3(a+b)EI} = \frac{Wa^2b^2}{3EIl} \qquad \text{...(8·11)}$$

If the concentrated load W is acting at mid-span, then

$$a = b = \frac{l}{2}$$

$$\therefore \quad (y)_{x=l/2} = \frac{Wl^3}{48EI} \qquad \text{...(8·12)}$$

To determine the maximum deflection,

$$\frac{dy}{dx} = 0$$

$$\therefore \quad -\frac{Wbx^2}{6(a+b)} + \frac{W(x-a)^2}{2} + \frac{Wab}{6}\left(\frac{a+2b}{a+b}\right) = 0$$

For x to be more than a, the second bracket becomes negative and is neglected.

$$\therefore \quad x^2 = \frac{a(a+2b)}{3}$$

or

$$x = \sqrt{\frac{a(a+2b)}{3}}$$

$$\therefore \quad EIy_{max} = \frac{Wab(a+2b)}{9(a+b)}\left\{\frac{a}{3}(a+2b)\right\}^{1/2} \qquad \text{...(8·13)}$$

At $x = 0$

$$EI\left(\frac{dy}{dx}\right)_{x=0} = \frac{Wab}{6}\left(\frac{a+2b}{a+b}\right)$$

∴ $$\theta_A = \frac{Wab}{6EI}\left(\frac{a+2b}{a+b}\right) \quad ...(8\cdot14)$$

For $$a = b = \frac{l}{2}$$

$$\theta_A = \frac{Wl^2}{16El} \quad ...(8\cdot15)$$

At $$x = l$$

$$EI\left(\frac{dy}{dx}\right)_{x=l} = -\frac{Wbl}{2} + \frac{Wb^2}{2} + \frac{Wab}{6}\left(\frac{a+2b}{a+b}\right)$$

$$= \frac{Wb}{2}\left[-l + b + \frac{a}{3}\left(\frac{a+2b}{l}\right)\right] = -\frac{Wab}{6}\left(\frac{2a+b}{a+b}\right)$$

∴ $$\theta_B = \frac{Wab}{6EI}\left(\frac{2a+b}{a+b}\right) \quad ...(8\cdot16)$$

For $$a = b = \frac{l}{2}$$

$$\theta_B = \frac{Wl^2}{16EI} \quad ...(8\cdot17)$$

8·4·1·2 Uniformly distributed load on a simply supported beam. Consider a beam AB of length ℓ, simply supported at the ends and carrying a uniformly distributed load of intensity w per unit length as shown in Fig. 8·6. Taking a section at a distance x from left end A, we have

$$M_x = R_A \times x - w\frac{x^2}{2} \quad ...(a)$$

Now $$R_A = R_B = \frac{wl}{2}$$

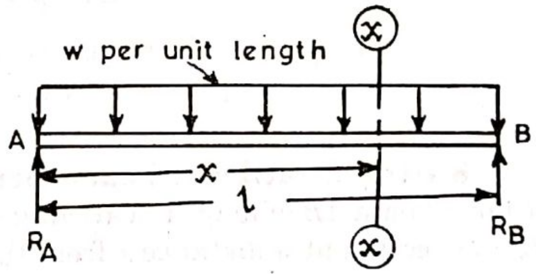

Fig. 8·6

Differential equation of bending is

$$EI\frac{d^2y}{dx^2} = -M_x$$

$$= -R_A \times x + \frac{wx^2}{2} = -\frac{wlx}{2} + \frac{wx^2}{2} \quad ...(b)$$

Integrating, we get

$$EI\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + C_1 \quad ...(c)$$

where C_1 is a constant of integration.

Integrating again, we get

$$EIy = -\frac{wlx^3}{12} + \frac{wx^4}{24} + C_1x + C_2 \quad ...(d)$$

Now at $$x = 0,\ y = 0$$

∴ $$C_2 = 0$$

Also at $x = l, y = 0$

$$\therefore \quad 0 = -\frac{wl^3}{12} + \frac{wl^4}{24} + C_1 l$$

$$\therefore \quad C_1 = \frac{wl^3}{24}$$

Substituting in Eq. (*d*), we get

$$EI\,y = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3x}{24} \quad ...(8\cdot18)$$

At $x = \frac{l}{2}$ the deflection is maximum

$$EI\,y_{max} = -\frac{wl^4}{96} + \frac{wl^4}{384} + \frac{wl^4x}{48} = \frac{5wl^4}{384}$$

$$y_{max} = \frac{5wl^4}{384\,EI} \quad ...(8\cdot19)$$

Similarly Eq. (*c*) becomes :

$$EI\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^2}{24} \quad ...(8\cdot20)$$

At $x = 0$

$$EI\theta_A = \frac{wl^2}{24}$$

$$\theta_A = \frac{wl^3}{24\,EI} \quad ...(8\cdot21)$$

At $x = l$

$$EI\theta_B = -\frac{wl^3}{4} + \frac{wl^3}{6} + \frac{wl^3}{24} = -\frac{wl^3}{24}$$

$$\theta_B = \frac{wl^3}{24\,EI} \quad ...(8\cdot22)$$

8·4·1·3 Cantilever beam carrying a concentrated load at the free end. Consider a cantilever beam AB of length ℓ carrying a concentrated load W at the free end as shown in Fig. 8·7. Taking a section at a distance x from the free end, we have

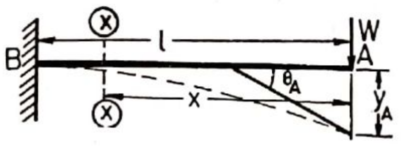

Fig. 8·7 Cantilever carrying concentrated load at free end.

$M_x = -Wx$ being a hogging moment. ...(*a*)

Thus differential equation of bending becomes,

$$EI\frac{d^2y}{dx^2} = -M_x = Wx \quad ...(a)$$

Integrating, we get

$$EI\frac{dy}{dx} = \frac{Wx^2}{2} + C_1 \quad ...(b)$$

where C_1 is a constant of integration.

Integrating again, we get $EIy = \frac{Wx^2}{6} + C_1x + C_2$ ...(*c*)

where C_2 is another constant of integration.

At $$x = l,\ y = 0,\ \frac{dy}{dx} = 0$$

$$\therefore \quad C_1 = -\frac{Wl^2}{2}$$

$$C_2 = -\frac{Wl^3}{3}$$

Hence Eqs. (*b*) and (*c*) becomes respectively

$$EI\frac{dy}{dx} = \frac{Wx^2}{2} - \frac{Wl^2}{2} \quad ...(8\cdot23)$$

$$EIy = \frac{Wx^3}{6} - \frac{Wl^2}{2}x + \frac{Wl^3}{3} \quad ...(8\cdot24)$$

At $$x = 0$$

$$\theta_A = -\frac{Wl^2}{2EI} \quad ...(8\cdot25)$$

$$y_A = \frac{Wl^3}{3EI} \quad ...(8\cdot26)$$

-ve sign indicates that the slope is convex upwards.

8·4·1·4 Cantilever beam carrying a uniformly distributed load over the whole span. Consider a cantilever *AB* of length l and carrying uniformly distributed load of intensity w per unit length as shown in Fig. 8·8. Consider a section at a distance x from the free end *A*. Then at *x–x*,

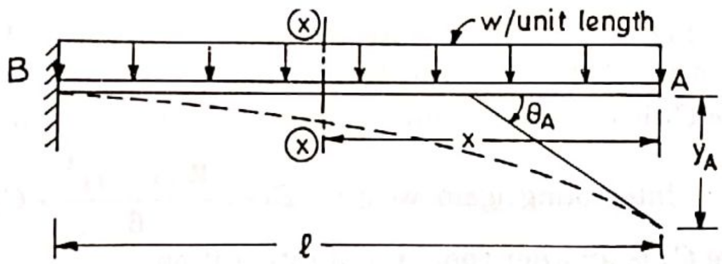

Fig. 8·8 Cantilever carrying u.d.l. over whole span.

$$M_x = -\frac{wx^2}{2}$$

being a hogging moment. ...(*a*)

The differential equation for bending becomes

$$EI\frac{d^2y}{dx^2} = -M_x = \frac{wx^2}{2}$$

Integrating, we get $$EI\frac{dy}{dx} = \frac{wx^3}{6} + C_1 \quad ...(b)$$

where C_1 is a constant of integration.

Integrating again, we get $$EIy = \frac{wx^4}{24} + C_1x + C_2 \quad ...(c)$$

where C_2 is another constant of integration.

Now at $$x = l,\ y = 0$$

and $$\frac{dy}{dx} = 0$$

$$\therefore \quad C_1 = -\frac{wl^3}{6}$$

$$C_2 = -\frac{wl^4}{24} + \frac{wl^4}{6} = \frac{wl^4}{8}$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{wx^3}{6} - \frac{wl^3}{6} \quad \text{...(8·27)}$$

$$EIy = \frac{wx^4}{24} - \frac{wl^3}{6} + \frac{wl^4}{8} \quad \text{...(8·28)}$$

At $x = 0$

$$\theta_A = -\frac{wl^3}{6EI} \quad \text{...(8·29)}$$

$$y_A = \frac{wl^4}{8EI} \quad \text{...(8·30)}$$

– ve sign indicates that slope is convex upwards.

8·4·1·5 Cantilever beam carrying a concentrated load not at the free end. Consider a cantilever beam AB of length l and carrying a concentrated load W at a distance a from the free end A. Taking a section at a distance x from the free end, the bending moment at x–x is (Fig. 8·9)

$$M_x = -W(x-a) \quad \text{...(a)}$$

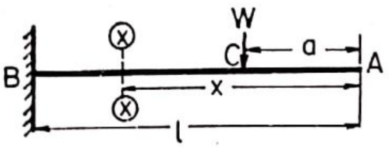

Fig. 8·9 Cantilever carrying concentrated load any where.

$$EI\frac{d^2y}{dx^2} = -M_x = W(x-a)$$

Integrating we get

$$EI\frac{dy}{dx} = \frac{W(x-a)^2}{2} + C_1 \quad \text{...(b)}$$

where C_1 is constant of integration.

Integrating again, we get $$EIy = \frac{W(x-a)^3}{6} + C_1x + C_2 \quad \text{...(c)}$$

where C_2 is another constant of integration.

At $x = l,\ y = 0$

$$\frac{dy}{dx} = 0$$

$$\therefore \quad C_1 = -\frac{W(l-a)^2}{2}$$

and

$$0 = \frac{W(l-a)^3}{6} - \frac{W(l-a)^2 l}{2} + C_2$$

$$\therefore \quad C_2 = -\frac{W(l-a)^3}{6} + \frac{W(l-a)^2 l}{2}$$

$$= \frac{W(l-a)^2}{6}[-(l-a)+3l] = \frac{W(l-a)^2}{6}[2l+a]$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{W(x-a)^2}{2} - \frac{W(l-a)^2}{2} \quad \text{...(8·31)}$$

$$EIy = \frac{W(x-a)^3}{6} - \frac{W(l-a)^2}{2}x + \frac{W(l-a)^2}{6}(2l+a) \quad \text{...(8·32)}$$

At $\quad x = a$

$$EI\frac{dy}{dx} = -\frac{W(l-a)^2}{2}$$

$\therefore$
$$\frac{dy}{dx} = \theta_C = -\frac{W(l-a)^2}{2EI} \quad ...(8\cdot33)$$

$$EIy = -\frac{W(l-a)^2 a}{2} + \frac{W(l-a)^2}{6}(2l+a)$$

$$= \frac{W(l-a)^2}{6}[-3a+2l+a] = \frac{W(l-a)^2}{6}(2l-2a)$$

$\therefore$
$$y = \frac{W(l-a)^3}{3EI} \quad ...(8\cdot34)$$

Negative sign for slope at C indicates that it is convex upwards.

8·4·1·6 Simply supported beam subjected to a pure couple. Consider a simply supported beam AB of length l and subjected to a couple M_0 at point C at a distance a from left end support A as shown in Fig. 8·10. Taking a section at a distance x from A,

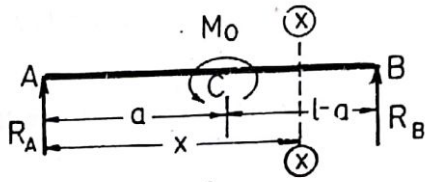

Fig. 8·10 Beam subjected to pure couple.

$$M_x = R_A \times x - M_0(x-a)^0 \quad ...(a)$$

where
$$R_A = \frac{M_0}{l},\ R_B = -\frac{M_0}{l}$$

Then
$$EI\frac{d^2y}{dx^2} = -M_x = \frac{M_0}{l}x + M_0(x-a)^0$$

Integrating, we get
$$EI\frac{dy}{dx} = -\frac{M_0}{l} \times \frac{x^2}{2} + M_0(x-a) + C_1 \quad ...(b)$$

where C_1 is a constant of integration.

Integrating again, we get
$$EIy = -\frac{M_0 x^3}{6l} + M_0\frac{(x-a)^2}{2} + C_1 x + C_2 \quad ...(c)$$

where C_2 is anaother constant of integration.

At $\quad x = 0,\ y = 0$

$\therefore \quad C_2 = 0$

At $\quad x = l,\ y = 0$

$\therefore$
$$0 = \frac{M_0 l^2}{6} + M_0\frac{(l-a)^2}{2} + C_1 l$$

$\therefore$
$$C_1 = \frac{M_0 l}{6} - M_0\frac{(l-a)^2}{2l}$$

$\therefore$
$$EI\frac{dy}{dx} = \frac{M_0 x^2}{2l} + M_0(x-a) + \frac{M_0 l}{6} - \frac{M_0(l-a)^2}{2l} \quad ...(8\cdot35)$$

At $\quad x = 0$

$$\theta_A = 0 = \frac{M_0 l}{6EI} - \frac{M_0(l-a)^2}{2EIl} \quad ...(8\cdot36)$$

At $x = a$

$$EI\theta_C = \frac{M_0 l}{2l} + \frac{M_0 l}{6} - \frac{M_0(l-a)^2}{2l}$$

$$= -\frac{M_0 a^2}{l} - \frac{M_0 l}{3} + M_0 a \quad \text{...(8·37)}$$

$$EIy = -\frac{M_0 x^3}{6l} + \frac{M_0 (x-a)^2}{2} + \frac{M_0 lx}{6} - \frac{M_0 (l-a)^2 x}{2l} \quad \text{...(8·38)}$$

At $x = a$

$$EIy_C = \frac{M_0 a^3}{6l} + \frac{M_0 la}{6} - \frac{M_0 (l-a)^2 a}{2l}$$

$$= -\frac{2}{3}\frac{M_0 a^3}{l} - \frac{M_0 la}{3} + M_0 a^2 \quad \text{...(8·39)}$$

If $a = \frac{l}{2}$, then

$$\left.\begin{aligned} \theta_C &= -\frac{M_0 l}{12EI} \\ y_C &= 0 \end{aligned}\right\} \quad \text{...(8·40)}$$

8·4·1·7 Beams of varying cross-section. For a beam of varying cross-section and made of different materials, EI should be taken inside the integration sign. For example,

$$EI\frac{d^2y}{dx^2} = -M_x$$

$$\frac{d^2y}{dx^2} = -\frac{M_x}{EI}$$

$$\therefore \quad \frac{dy}{dx} = -\int \frac{M_x}{EI}\,dx$$

Consider a beam made of two materials and different cross-section as shown in Fig. 8·11. Then

$$M_x = R_A \times x - W_1 (x-a) - W_2 (x-b)$$

$$R_A (l_1 + l_2) = W_1 (l_1 + l_2 - a) + W_2 (l_1 + l_2 - b)$$

$$R_A = W_1\left(1 - \frac{a}{l_1 + l_2}\right) + W_2\left(1 - \frac{b}{l_1 + l_2}\right)$$

$$R_B = (W_1 + W_2) - R_A$$

$$EI\frac{d^2y}{dx^2} = -M_x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{M_x}{EI}$$

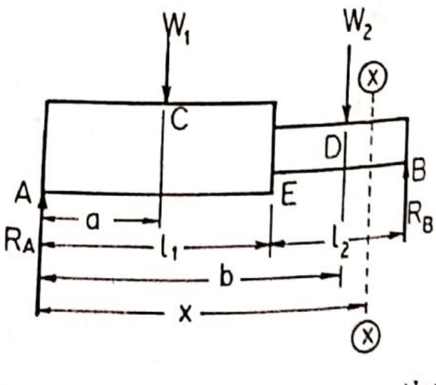

Fig. 8·11. Beam of varying cross-section.

$$= -\frac{R_A \times x}{EI} + \frac{W_1 (x-a)}{EI} + \frac{W_2 (x-b)}{EI}$$

Integrating, we get

$$\frac{dy}{dx}=\int_0^{l_1}\left\{\frac{-R_A\times x}{E_1I_1}+\frac{W_1(x-a)}{E_1I_1}+\frac{W_2(x-b)}{E_1I_1}\right\}dx$$

$$+\int_{l_1}^{l_2}\left\{\frac{-R_A\times x}{E_2I_2}+\frac{W_1(x-a)}{E_2I_2}+\frac{W_2(x-b)}{E_2I_1}\right\}dx+C_1$$

Then

$$y=\int_0^{l_1}\int_0^{l_1}\left\{\frac{-R_A\times x}{E_1I_1}+\frac{W_1(x-a)}{E_1I_1}+\frac{W_2(x-b)}{E_1I_1}\right\}dx.dx$$

$$+\int_{l_1}^{l_2}\int_{l_1}^{l_2}\left\{\frac{-R_A\times x}{E_2I_2}+\frac{W_1(x-a)}{E_2I_2}+\frac{W_2(x-b)}{E_2I_2}\right\}dx.dx+C_1x+C_2$$

8·4·1·8 Propped Beams and Cantilevers. In order to determine the known reaction at the prop, the deflection at the prop may be thought of to be composed of two parts.

(*i*) Downward deflection at the prop location due to the external applied loads

(*ii*) Upward deflection at the prop location due to the reaction at the prop.

But the deflection at the prop is zero.

Hence the algebraic sum of the above two deflections should be equated to zero to determine the unknown reaction at the prop.

8·4·1·9 Sinking of Prop. If the prop is of the sinking type or an elastic prop and sinks by an amount δ, then the resultant deflection at the prop location should be equated to δ.

Example 8·1 *A horizontal girder of steel having uniform section is 14 m long and is simply supported at its ends. It carries concentrated loads of 120 kN and 80 kN at two points 3 m and 4·5 m from the two ends respectively. I for the section of the girder is 16 × 10⁴ cm⁴ and E for steel is 210 GPa. Calculate the deflection of the girder at points under the two loads.*

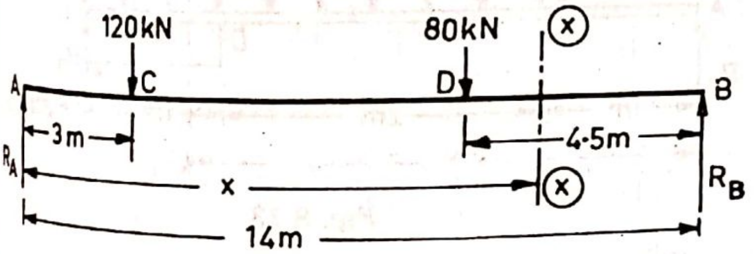

Fig. 8·12

Solution. Taking moments about B, we get

$$R_A\times 14=120\times 11+80\times 4{\cdot}5$$
$$=1680$$
$$R_A=120\text{ kN}$$

Taking a section at x–x as shown in Fig. 8·12,

$$M_x=R_A\times x-120\,(x-3)-80\,(x-9{\cdot}5)$$

∴
$$EI\frac{d^2y}{dx^2}=-120x+120\,(x-3)+80\,(x-9{\cdot}5)$$

Integrating, we get

$$EIy=-20x^3+20\,(x-3)^3+\frac{40}{3}\,(x-9{\cdot}5)^3+C_1x+C_2$$

As $x=0,\ y=0$

∴ $C_2=0$

At $x=14\text{ m},\ y=0$

$$0=-20\times 14^3+20\,(14-3)^3+\frac{40}{3}(14-9{\cdot}5)^3+14C_1$$

$$= -54880 + 26620 + 1215 + 14C_1 = -27045 + 14C_1$$

$$C_1 = 1931{\cdot}8 \text{ kN-m}^3$$

$\therefore$ $$EIy = -20x^3 + 20(x-3)^3 + \frac{40}{3}(x-9{\cdot}5)^3 + 1931{\cdot}8x$$

At $x = 3$ m

$$EIy = -540 + 20(3-3)^3 + \frac{40}{3}(3-9{\cdot}5)^3 + 5795{\cdot}4$$

$$= -540 + 5795{\cdot}4 = 5255{\cdot}4$$

$$y = \frac{5255{\cdot}4 \times 10^3}{210 \times 10^3 \times 16 \times 10^{-4}} = 1{\cdot}564 \times 10^{-2} \text{ m}$$

Note that the third bracket being negative is neglected.

At $x = 9{\cdot}5$ m,

$$EIy = -20(9{\cdot}5)^3 + 20(9{\cdot}5-3)^3 + \frac{40}{3}(9{\cdot}5-9{\cdot}5)^3 + 1931{\cdot}8 \times 9{\cdot}5$$

$$= -17147{\cdot}5 + 5492{\cdot}5 + 18351{\cdot}6 = 6696{\cdot}6 \text{ kN-m}^3$$

$\therefore$ $$y = \frac{6696{\cdot}6 \times 10^3}{210 \times 10^9 \times 10^{-4} \times 16} = 1{\cdot}994 \times 10^{-2} \text{ m}$$

Example 8·2 *A beam AB simply supported at the ends is 4 m long. It carries a uniformly distributed load of intensity 20 kN/m over a length of 2 m starting at a distance of 1 m from left end support together with a concentrated load 40 kN at a distance of 3 m from the left end support. Calculate the deflection at the centre, if E = 210 GPa, 1 = 9600 cm⁴.*

Solution. Consider a section at a distance x from the left end support as shown in Fig. 8·10. The distributed load does not extend upto the right end support. So we extend this load upto the right end and apply equal and opposite load over the portion DB.

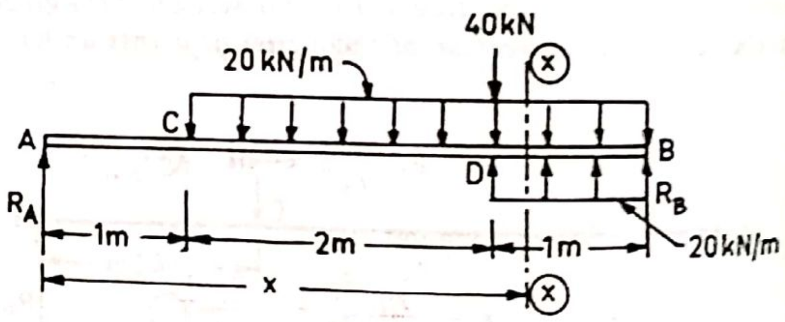

Fig. 8·13

Taking moments about B, we get

$$R_A \times 4 = 40 \times 1 + 20 \times 2 \times 2 = 120$$

$\therefore$ $$R_A = 30 \text{ kN}$$

$$M_x = 30x - 20(x-1)\cdot\frac{(x-1)}{2} + 20(x-3)\cdot\frac{(x-3)}{2} - 40(x-3)$$

$$= 30x - 10(x-1)^2 + 10(x-3)^2 - 40(x-3)$$

Now $$EI\frac{d^2y}{dx^2} = -M_x$$

$$= -30x + 10(x-1)^2 - 10(x-3)^2 + 40(x-3)$$

Integrating twice, we get

$$EI\frac{dy}{dx} = -15x^2 + \frac{10(x-1)^3}{3} - \frac{10(x-3)^3}{3} + 20(x-3)^2 + C_1$$

$$EIy = -5x^3 + \frac{5(x-1)^4}{6} - \frac{5(x-3)^4}{6} + \frac{20(x-3)^3}{3} + C_1x + C_2$$

At $x = 0, \quad y = 0$

$\therefore$ $C_2 = 0$

At $x = 4$ m, $\quad y = 0$

$\therefore$ $0 = -320 + 67{\cdot}5 - 0{\cdot}83 + 6{\cdot}67 + 4C_1 = -246{\cdot}46 + 4C_1$

$\therefore$ $C_1 = 61{\cdot}61$ kN-m³

$$EIy = -5x^3 + \frac{5(x-1)^4}{6} - \frac{5(x-3)^4}{6} + \frac{20}{3}(x-3)^3 + 61{\cdot}61\,x$$

At mid-span, $x = 2$ m

$$EIy = -40 + 0{\cdot}833 \times 123{\cdot}22$$

$$y = \frac{84{\cdot}053 \times 10^3}{210 \times 10^9 \times 9600 \times 10^{-8}} = 4{\cdot}169 \times 10^{-3} \text{ m}$$