Simple Bending of Beams

7·1 Introduction

In this chapter, we shall study the bending of prismatic beams of symmetrical cross-section under the action of bending moments only. The distribution of horizontal shear in beams will also be discussed in detail. Before actually studying the theory of simple bending, the knowledge of some of the definitions is essential.

Definitions

Centroid. The centroid of a figure is that point in the figure at which all the area may be assumed to be concentrated. If a figure be sub-divided into rectangular areas, A_1, A_2,...and the perpendicular distances of the centroids of these areas from two chosen coordinate axes x and y are y_1, y_2 ...and x_1, x_2...respectively then the coordinates of the centroid $(\bar{x}, \bar{y})$ of the area of the whole figure are given by :

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2 + \ldots}{A_1 + A_2 + \ldots} = \frac{\Sigma A_i x_i}{\Sigma A_i} = \frac{\Sigma A_i x_i}{A} \quad \ldots(7\cdot1)$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + \ldots}{A_1 + A_2 + \ldots} = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{\Sigma A_i y_i}{A} \quad \ldots(7\cdot2)$$

where A = total area of the figure.

Moment of inertia. The moment of inertia or the second moment of the area about a given axis is the product of the element of area and the square of the distance of the centroid from the axis.

Thus $I_x = \Sigma y^2 \, dA$

and $I_y = \Sigma x^2 \, dA$

or $I = AK^2 \quad \ldots(7\cdot3)$

where A = area of the figure

K = radius of gyration.

Parallel axis theorem. The moment of inertia of a figure about any axis in its plane is equal to the total sum of the moment of inertia of the figure about a parallel axis through the centroid of the figure plus the product of the total area of the figure and the square of the distance between the parallel centroidal axis and the axes of reference. For Example

$$\left.\begin{aligned} I_{xx} &= I_{\bar{x}\bar{x}} + A\bar{y}^2 \\ I_{yy} &= I_{\bar{y}\bar{y}} + A\bar{x}^2 \end{aligned}\right\} \quad \ldots(7\cdot4)$$

Neutral axis. Neutral axis of a beam is the axis at which the bending stress is zero.

Simple bending. When the load producing bending lies in the centroidal plane such that bending is not accompanied by torsion then the bending is said to be simple bending.

Pure bending. When a beam is subjected to such a system of bending loads so that the shear force in the beam is zero then the beam is said to be subjected to pure bending. In such a case the bending moment shall be constant in the beam.

7·3 Theory of Simple Bending

When a beam is bent due to the application of a constant bending moment, without being subjected to shear, it is said to be in a state of simple bending. In simple bending the plane of transverse loads and the centroidal plane coincide. The theory of simple bending was developed by Galelio, Bernoulli and St. Venant and is also some times called the Bernoulli's theory of simple bending. The following assumptions are made in this theory :

1. The material is assumed to be homogeneous, perfectly elastic and isotropic.
2. A transverse section of a beam, which is plane before bending, will remain a plane after bending.
3. The radius of curvature of the beam before bending is very large in comparison to the transverse dimensions of the beam.
4. The resultant push or pull across a tranverse section of the beam is zero.
5. The elastic limit is nowhere exceeded.
6. Young's modulus for the material is same in tension and compression.
7. The transverse section of the beam is symmetrical about an axis passing through the centroid of the section and parallel to the plane of bending.

Consider the portion of a beam subjected to simple bending as shown in Fig. 7·1(*a*). In the unstrained state, let *GH* be a portion of a fibre at a distance *y* from the centroidal axis *KL*, its length being determined by the two transverse parallel planes *AD* and *BC*. After bending, the planes *AD* and *BC* assume the positions $A_1 D_1$ and $B_1 C_1$ respectively as shown in Fig. 7·1 (*b*), being inclined at an angle θ and intersecting at the point *O*, the centre of curvature. Let *R* be the radius of the centroidal surface $E_1 F_1$ so that the radius of surface $G_1 H_1$ is $(R + y)$.

Now $$\frac{G_1 H_1}{EF} = \frac{(R+y)\,\theta}{R\theta} = \frac{R+y}{R}$$

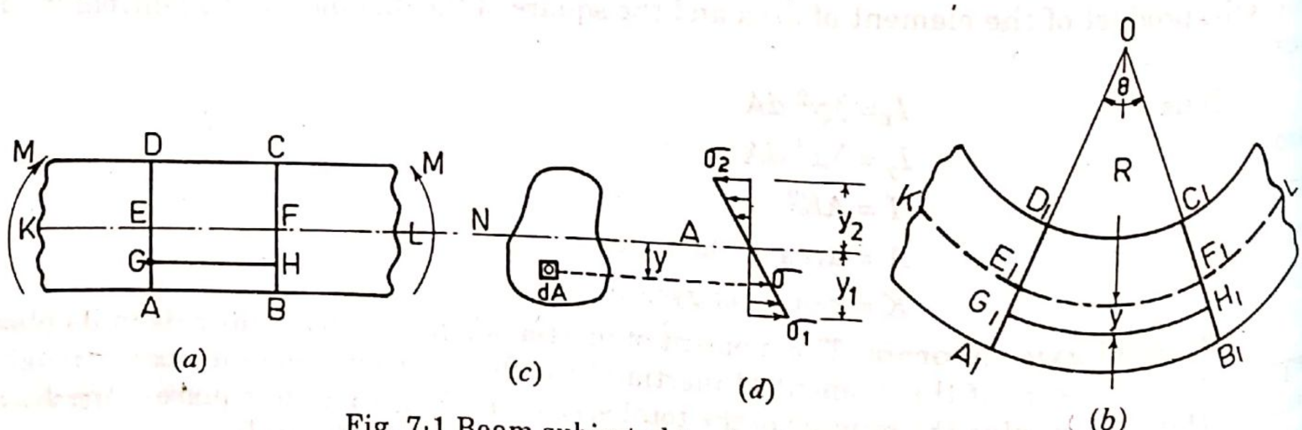


Fig. 7·1 Beam subjected to simple bending.

Strain in the fibres at *GH* is, $\varepsilon = \dfrac{G_1 H_1 - GH}{GH}$

$$= \frac{G_1 H_1 - EF}{EF} = \frac{G_1 H_1}{EF} - 1 = \frac{R+y}{R} - 1$$

$$\varepsilon = \frac{y}{R}$$

or

If σ = intensity of stress in the fibres, then

$$\sigma = E\varepsilon$$

or

$$\sigma = \frac{Ey}{R}$$

or

$$\frac{\sigma}{y} = \frac{E}{R} \quad ...(a)$$

For a given load on a beam, $\frac{E}{R}$ is constant (say = k) then, $\sigma = ky$, *i.e.* the stress in the fibres of a beam at any point in the cross-section is proportional to its distance from the centroidal axis. Thus the bending stress will be maximum at the boundary of the beam which is at the greatest distance from the centroidal axis.

In order to locate the position of the neutral axis, consider an element of area dA at a distance y from the centroidal axis [Fig. 7·1 (c)]. Total force on the element is then equal to,

$$dF = \sigma . dA$$

But

$$\frac{\sigma}{y} = \frac{\sigma_1}{y_1}$$

$$\therefore \quad \sigma = \sigma_1 . \frac{y}{y_1}$$

Hence

$$dF = \sigma_1 . \frac{y}{y_1} . dA$$

Total tensile force on the transverse section below the centroidal axis is then (Fig. 7·1 d).

$$F_1 = \Sigma\, \sigma_1 . \frac{y}{y_1} . dA = \frac{\sigma_1}{y_1} \Sigma\, ydA$$

If the elementary area is chosen on the upper side, then the total compressive force on the transverse section above the centroidal axis will be,

$$F_2 = \frac{\sigma_2}{y_2} \Sigma\, ydA$$

For equilibrium of the beam,

$$F_1 = F_2$$

$$\therefore \quad \frac{\sigma_1}{y_1} = \frac{\sigma_2}{y_2}$$

Further, since there is no resultant force across any transverse cross-section, therefore, $\Sigma ydA = 0$, *i.e.* the first moment of the area about the centroidal axis is zero which is possible only if the neutral axis passes through the centroidal axis. Thus in case of simple bending, the neutral axis passes through the centroid of the section.

Now the moment of the force acting on the elementary area dA about the neutral axis is,

$$dM = \frac{\sigma_1}{y_1} y^2 dA$$

The total moment of all the forces acting on various elements composing the cross-section forms a couple which is equal to the bending moment M. This total moment is called the *moment of resistance.*

$$M = \frac{\sigma_1}{y_1} \Sigma\, y^2 dA$$

$\therefore$

Now $\Sigma y^2 dA = I$, the moment of inertia of the cross-section about the neutral axis.

$\therefore$ $$M = \frac{\sigma_1}{y_1} I$$

Hence $$\frac{M}{I} = \frac{\sigma_1}{y_1} = \frac{\sigma}{y} \quad ...(b)$$

Combining Eqs. (*a*) and (*b*), we get

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \quad ...(7{\cdot}5)$$

This is the well-know bending formula.

Also $$M = \sigma \frac{I}{y}$$

or $$M = \sigma z$$

where $$z = \frac{I}{y} \quad ...(7{\cdot}6)$$

is called the *section modulus.*

Example 7·1 *A rectangular beam 6 cm × 4 cm is 2 m long and is simply supported at the ends. It carries a load 1 kN at mid-span. Determine the maximum bending stress induced in the beam.*

Solution. Maximum bending moment

$$M = \frac{Wl}{4} = \frac{1 \times 2}{4} = 0{\cdot}5 \text{ kN-m}$$

$$I = \frac{bh^3}{12} = \frac{6 \times 4^3 \times 10^{-8}}{12} = 32 \times 10^{-8} \text{ m}^4$$

Maximum bending stress will occur at $y = 2$ cm.

Now $$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{M}{I} . y$$

$$= \frac{0{\cdot}5 \times 10^3 \times 2 \times 10^{-2}}{32 \times 10^{-8}} = 31{\cdot}25 \text{ MPa}$$

Exampple 7·2 *The cross-section of a cast-iron beam is shown in Fig. 7·2 (a). This beam is simply supported at the ends and carries a uniformly distributed load of 20 kN/m. If the span of the beam is 3 m, determine the maximum tensile and compressive stresses in the beam.*

Solution. Taking moments about the bottom edge of the beam,

$$(10 \times 2 + 2 \times 10 + 20 \times 3)\bar{y} = 10 \times 2 \times 14 + 2 \times 10 \times 8 + 20 \times 3 \times 1{\cdot}5$$

$$(20 + 20 + 60)\bar{y} = 280 + 160 + 90$$

$$100\, \bar{y} = 530$$

$$\bar{y} = 5{\cdot}3 \text{ cm}$$

$$I = \frac{10 \times 2^3}{12} + 10 \times 2(14 - 5{\cdot}3)^2 + \frac{2 \times 10^3}{12} + 2 \times 10(8 - 5{\cdot}3)^2 + \frac{20 \times 3^3}{12} + 20 \times 3(5{\cdot}3 - 1{\cdot}5)^2$$

$$= 6{\cdot}667 + 1513{\cdot}8 + 165{\cdot}667 + 145{\cdot}8 + 866{\cdot}4$$

$$= 2744{\cdot}334 \text{ cm}^4 = 2744{\cdot}334 \times 10^{-8} \text{ m}^4$$

Now $M = \frac{wl^2}{8} = \frac{20 \times (3)^2}{8}$

$= 22{\cdot}5$ kN-m

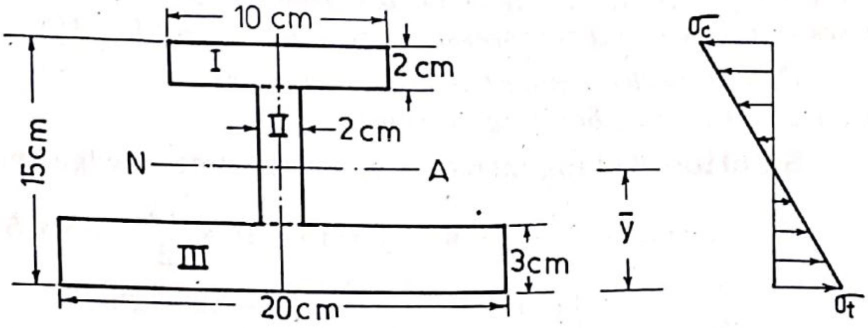
$$\frac{M}{I} = \frac{\sigma}{y}$$

Maximum compressive bending stress,

$$\sigma_c = \frac{M}{I}(15 - 5{\cdot}3)10^{-2}$$

$$= \frac{22{\cdot}5 \times 10^3 \times 9{\cdot}7 \times 10^{-2}}{2744{\cdot}334 \times 10^{-8}}$$

$= 79{\cdot}527$ MPa



(a) (b) Stress-distribution.

Fig. 7·2

Maximum tensile bending stress,

$$\sigma_t = \frac{M}{I} \times 5{\cdot}3 \times 10^{-2}$$

$$= \frac{22{\cdot}5 \times 10^3 \times 5{\cdot}3 \times 10^{-2}}{2744{\cdot}334 \times 10^{-8}} = 43{\cdot}453 \text{ MPa}$$

Example 7·3 *Two wooden planks 5 cm × 15 cm each are connected together to form a cross-section of a beam as shown in Fig. 7·3 (a). If a bending moment of 3400 N-m is applied around the horizontal neutral axis, find the stresses at the extreme fibres of the cross-section. Also calculate the total tensile force on the cross-section.*

Solution. Taking moments about the bottom edge, we have

$$(15 \times 5 + 15 \times 5)\ \bar{y} = 15 \times 5 \times 17{\cdot}5 + 15 \times 5 \times 7{\cdot}5$$

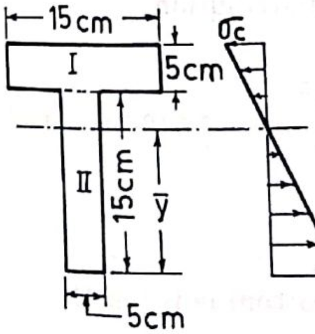
$$150\ \bar{y} = 1875$$

$$\bar{y} = 12{\cdot}5 \text{ cm}$$

$$I = \frac{15 \times 5^3}{12} + 15 \times 5 \times (17{\cdot}5 - 12{\cdot}5)^2 + \frac{5 \times 15^3}{12} + 5 \times 15\ (12{\cdot}5 - 7{\cdot}5)^2$$

$$= 156{\cdot}25 + 1875 + 1406{\cdot}25 + 1875$$

$$= 5312{\cdot}5 \text{ cm}^4 = 5312{\cdot}5 \times 10^{-8} \text{ m}^4$$



(a) (b) Stress distribution.

Fig. 7·3

$$\sigma_t = \frac{M}{I} \times 12{\cdot}5 \times 10^{-2} = \frac{3400 \times 12{\cdot}5 \times 10^{-2}}{5312{\cdot}5 \times 10^{-8}} = 8 \text{ MPa}$$

$$\sigma_c = \frac{M}{I} \times 7{\cdot}5 \times 10^{-2} = \frac{3400 \times 7{\cdot}5 \times 10^{-2}}{5312{\cdot}5 \times 10^{-8}} = 4{\cdot}8 \text{ MPa}$$

Total tensile force on the section

$$= \int_0^{12\cdot5} \left(\frac{\sigma}{12\cdot5} \times y \right) b\,dy = \frac{8 \times 10^6}{12\cdot5} \times 5 \int_0^{12\cdot5} y\,dy$$

$$= 3\cdot2 \times 10^6 \left| \frac{y^2}{2} \right|_0^{12\cdot5} = 1\cdot6 \times 10^6 \, (12\cdot5)^2 \times 10^{-4} = 25 \text{ kN}$$

Example 7·4 *The cross-section of a cast iron machine element used as a beam is shown in Fig. 7·4 (a). The beam resists bending moments about the horizontal neutral axis. The permissible stresses in tension and compression are to be 22 and 88 MPa respectively.*

Calculate the moment of resistance of the section about the horizontal neutral axis for both positive and negative bending moments.

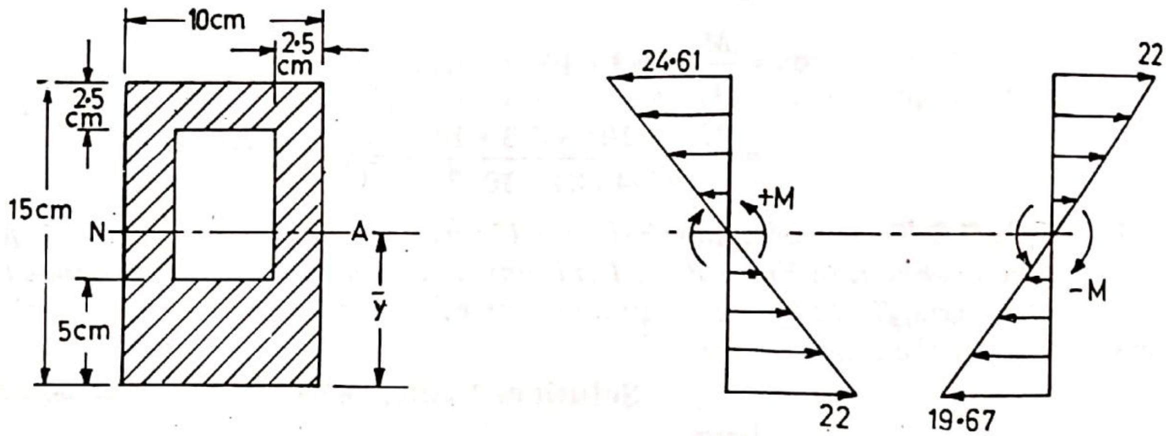
Solution. Taking moments about the bottom edge, we get

$$(15 \times 10 - 7\cdot5 \times 5)\, \bar{y} = 15 \times 10 \times \frac{15}{2} - 7\cdot5 \times 5 \times 8\cdot75$$

$$(150 - 37\cdot5)\, \bar{y} = 1125 - 328\cdot125$$

$$112\cdot5\, \bar{y} = 796\cdot875$$

$$\bar{y} = 7\cdot08 \text{ cm}$$



(a) (b) Stress-distribution.

Fig. 7·4

$$I = \frac{10 \times 15^3}{12} + 10 \times 15\,(7\cdot5 - 7\cdot08)^2 - \frac{5 \times (7\cdot5)^3}{12} - 5 \times 7\cdot5\,(8\cdot75 - 7\cdot08)^2$$

$$= 2812\cdot5 + 26\cdot46 - 115\cdot78 - 104\cdot58$$

$$= 2558\cdot60 \text{ cm}^4 = 2558\cdot6 \times 10^{-8} \text{ m}^4$$

For positive moment, stress at the top is compressive and at the bottom is is tensile.

Now $\sigma_t = 22$ MPa

$\therefore$ $\sigma_c = \frac{22}{7\cdot08} \times 7\cdot92 = 24\cdot61$ MPa at the top

$$M = \frac{\sigma_t}{y} \times I = \frac{22 \times 10^6 \times 2558\cdot6 \times 10^{-8}}{7\cdot08 \times 10^{-2}} = 7950\cdot5 \text{ N-m}$$

For negative moment, stress at the top will be tensile,

$\therefore$ Stress at the bottom, $\sigma_c = \frac{22}{7\cdot92} \times 7\cdot08 = 19\cdot67$ MPa

$$\therefore \quad M = \frac{22 \times 10^6 \times 2558 \cdot 6 \times 10^{-8}}{7 \cdot 92 \times 10^{-2}} = 7107 \cdot 2 \text{ N-m}$$

Example 7·5 *A wooden beam is 8 cm wide and 12 cm deep with a semi-circular groove of 2 cm radius planned out in the centre of each side. Calculate the maximum stress in the section when simply supported on a span of 3 m, loaded with a concentrated load of 450 N at a distance of 1 m from the one end and a uniformly distributed load of 500 N per metre run over the whole span.*

Solution. For the beam shown in Fig. 7·5 (*a*),

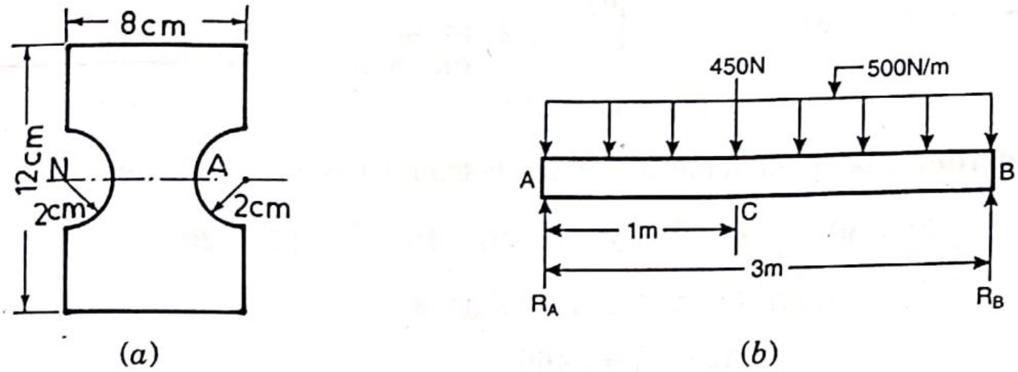


Fig. 7·5

$$I = \frac{8 \times 12^3}{12} - \frac{\pi}{64} \times 4^4 = 1152 - 12 \cdot 57$$

$$= 1139 \cdot 43 \text{ cm}^4 = 11389 \cdot 43 \times 10^{-8} \text{ m}^4$$

Now for the beam loaded as shown in Fig. 7·5 (*b*),

$$R_A \times 3 = 450 \times 2 + 500 \times 3 \times 1 \cdot 5 = 900 + 2250 = 3150$$

$$R_A = 1050 \text{ N}$$

$$R_A + R_B = 450 + 500 \times 3 = 1950 \text{ N}$$

$$R_B = 1950 - 1050 = 900 \text{ N}$$

$$M_x = R_A \times x - 450\,(x-1) - 500x \times \frac{x}{2}$$

$$= 1050x - 450x + 450 - 250x^2 = 600x + 450 - 250x^2$$

For bending moment to be maximum,

$$\frac{dM_x}{dx} = 0$$

$$600 - 500\,x = 0$$

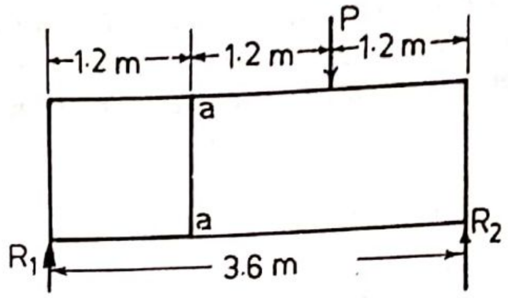
$$\therefore \quad x = 1 \cdot 2 \text{ m}$$

$$M_{max} = 600 \times 1 \cdot 2 + 450 - 250\,(1 \cdot 2)^2 = 720 + 450 - 360 = 810 \text{ N-m}$$

$$\therefore \quad \frac{\sigma}{y} = \frac{M}{I}$$

$$\therefore \quad \sigma = \frac{810 \times 6 \times 10^{-2}}{1139 \cdot 43 \times 10^{-8}} = 4 \cdot 265 \text{ MN/m}^2$$

Example 7·6 *The cross-section of a simply supported beam is shown in Fig. 7·6. The beam carries a load P = 10 kN as shown. Its self weight is 3·5 kN/m. Calculate the maximum normal stress at section a-a.*



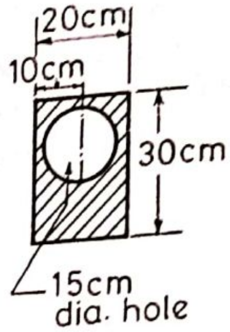


Fig. 7·6

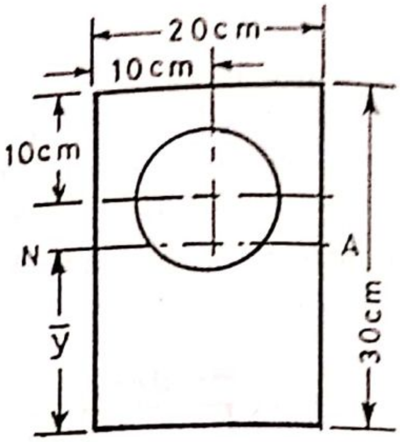


Fig. 7·7

Solution. Taking moment about the bottom edge (Fig. 7·7), we get

$$\left(20 \times 30 - \frac{\pi}{4} \times 15^2\right) \bar{y} = 20 \times 30 \times 15 - \frac{\pi}{4} \times 15^2 \times 20$$

$$(600–176·7)\ \bar{y} = 9000 - 3534$$

$$423·3\ \bar{y} = 5466$$

$$\bar{y} = 12·9 \text{ cm}$$

Moment of inertia about *N.A.*,

$$I = \frac{20 \times 30^3}{12} + 20 \times 30\,(15 - 12·9)^2 - \frac{\pi \times 15^4}{64} - \frac{\pi \times 15^2}{4}\,(20 - 12·9)^2$$

$$= 45{,}000 + 2646–2484·94 - 8907·44$$

$$= 36253·72 \text{ cm}^4 = 36253·72 \times 10^{-8} \text{ m}^4$$

Taking moments about R_2, we get

$$R_1 \times 3·6 = 10 \times 1·2 + 3·5 \times 3·6 \times 1·8 = 12 + 22·68 = 34·68$$

$$R_1 = \frac{34·68}{3·6} = 9·633 \text{ kN}$$

Bending moment at *a-a*,

$$M = 9·633 \times 1·2 - 3·50 \times 1·2 \times 0·6 = 11·56 - 2·52 = 9·04 \text{ kN-m}$$

$$\therefore \quad \sigma = \frac{M}{I} \times (30 - 12·9) \times 10^{-2} = \frac{9·04 \times 10^3 \times 17·1 \times 10^{-2}}{36253·72 \times 10^{-8}} = 4·264 \text{ MPa}$$