# * Concept of velocity potential ($\phi$) and stream function ($\psi$)

* Continuity equation for 3-D but in case of steady and incompressible

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

* Divergent $(\nabla) \cdot \vec{V}$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

delta function

$$\nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$\therefore \nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (u\hat{i} + v\hat{j} + w\hat{k})$$

Here $\Rightarrow$ $\left|\begin{matrix}\hat{i}\cdot\hat{i} = 1 \\ \hat{j}\cdot\hat{j} = 1 \\ \hat{k}\cdot\hat{k} = 1\end{matrix}\right|$ dot product.

$$\Rightarrow \boxed{\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial w}} \rightarrow$$ continuity equation for 3-D in case of steady & incompressible flows.

*[illegible]

## * Angular velocity of particle ($\Omega$)

$\Omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

ohm. omega

$\Omega = \frac{1}{2} \times (\nabla \times \vec{v}) = \frac{1}{2}$ of curl of velocity vector.

$$\Rightarrow \frac{1}{2} \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{pmatrix}$$

$$\Rightarrow \frac{1}{2} \left\{ \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\hat{i} + \hat{j}\left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) + \hat{k}\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right.$$

$$\therefore \quad \omega_x = \frac{1}{2}\left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)$$
$$\omega_y = \frac{1}{2}\left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right)$$
$$\omega_z = \frac{1}{2}\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

only for magnitude are considered

* if angular velocity $\to 0$ flow = Irrotational

$\omega_x = \omega_y = \omega_z = 0$ } Irrotational

$\Omega = 0$

* Vorticity $\Rightarrow$ ✗ नहीं पढ़ना है

$\xi = 2 \times \Omega = 2 \times$ angular velocity.

$= 2 \times \frac{1}{2} (\nabla \times \vec{v}) = (\nabla \times \vec{v})$

→ if vorticity $\to 0 \Rightarrow$ Irrotational flow

$$\xi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

→ Only for magnitude is consider.

# Velocity potential function and Stream function

## * Velocity potential function ⇒

→ It is defined as a Scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction.

→ It is ~~denoted~~ defined by $\phi$ (phi).

Mathematically,

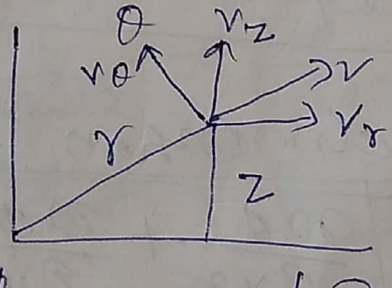
$$\phi = f(x, y, z) \text{ for steady flow}$$

Such that.

$$u = -\frac{\partial \phi}{\partial x} \quad v = -\frac{\partial \phi}{\partial y} \quad w = -\frac{\partial \phi}{\partial z}$$

where $u$, $v$, & $w$ are the components of velocity in $x$, $y$ & $z$ direction respectively.

* The velocity components in cylindrical polar co-ordinates in terms of velocity potential function are given by.




$$u_r = \frac{\partial \phi}{\partial r} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}.$$

where $u_r$ = velocity component in Radial direction (i.e in r-direction)

and $u_\theta$ = velocity component in tangential direction (i.e in θ-direction)

* for Steady & Incompressible flows.
continuity equation

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0}$$

→ for flows condition

$$u=-\frac{\partial \phi}{\partial x},\quad v=-\frac{\partial \phi}{\partial y},\quad w=-\frac{\partial \phi}{\partial z} \quad \text{— (I)}$$

Substituting the values of u. v & w. from eqn (I) we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right)=0$$

$$\text{or}\quad -\left[\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}\right]=0$$

$$\text{or}\quad \boxed{\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0} \quad \text{— (III)}$$

→ This equation is a laplace equation

* for two dimension case.

$$\boxed{\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0}$$

→ flow is possible only when it given $\phi$ (velocity potential function) follow laplace equation.

* Check for Rotation

$$\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\left.\begin{matrix} \omega_x = 0 \\ \omega_y = 0 \\ \omega_z = 0 \end{matrix}\right\} = \text{irrotational} \qquad u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}.$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right)$$

$$= \frac{1}{2}\left(-\frac{\partial^2 \phi}{\partial x \cdot \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right) = 0$$

Similarly

$$\cancel{\omega_y = \frac{1}{2}\left(\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial}\right.\right.}$$

$$\omega_y = \frac{1}{2}\left(-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right) = 0$$

$$\omega_x = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right] = 0$$

$$\therefore \left\{\begin{matrix} \omega_z = 0 \\ \omega_y = 0 \\ \omega_x = 0 \end{matrix}\right\}$$

⇒ ~~When rotational component are zero the flow is called irrotational~~

* If $\phi$ exist in any flow field then the flow must be irrotational.

* **Equipotential line** *

→ line joining points of equal velocity potential function called equipotential.

* **For 2-D, Steady, incompressible flow**

$$\phi = (x, y, t) \rightarrow (2\text{-D})$$

$$\phi = (x, y, z, t) - (3\text{-D}).$$

$$u = -\frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}.$$

in this $\phi = (x, y)$ $\phi$ = constant.

$$d\phi = 0$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} \cdot dy.$$

$$d\phi = -u\,dx - v\,dy = 0.$$

$$u\,dx = v\,dy.$$

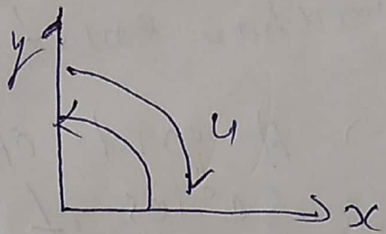
$$\text{slope} = \boxed{\frac{dy}{dx} = \tan\theta = -\frac{u}{v}} \rightarrow (2\text{-D})$$

# * Stream function ($\psi$)

→ it is defined only in (2-D) plane at a time.

→ it is a scalar function of space & time such that its partial derivative w.r.t any direction gives the velocity component at right angle in anticlockwise direction to this direction.

$$-u = \frac{\partial \psi}{\partial y}$$




$$\frac{\partial \psi}{\partial x} = v$$

$$\frac{\partial \psi}{\partial x} = v$$

$$\frac{\partial \psi}{\partial y} = -u$$

→ stream function is only valid for 2-D case at a time.

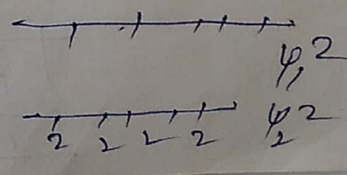
$$\psi = f(x, y)$$

$$\frac{\partial \psi}{\partial x} = v, \quad \frac{\partial \psi}{\partial y} = -u$$

## * Equistream line

→ The line joining constant stream function is equistream line

$$\psi = \text{constant}$$



$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} \cdot dy.$$

$d\psi = v.dx - u.dy = 0$, $\frac{dy}{dx} = \frac{v}{u}$ ✓

equipotential line → slope $= \frac{dy}{dx} = -\frac{u}{v} = m_1$

equistream line $\frac{dy}{dx} = \frac{v}{u} = m_2$

$m_1 \times m_2 = -\frac{u}{v} \times \frac{v}{u} = -1$

→ It means that the equipotential line and equistream line are orthogonal to each other in case of irrotational flow. where both potential function and stream function exist.

* **Flow Net** → A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. the flow net is an important tool in analysing two dimensional irrotational flow problem.

* **Relation between stream function and velocity potential function** →

we have $u = -\frac{\partial \phi}{\partial x}$ & $v = -\frac{\partial \phi}{\partial y}$.

& stream function $u = -\frac{\partial \psi}{\partial y}$ & $v = \frac{\partial \psi}{\partial x}$.

thus, $+\frac{\partial \phi}{\partial x} = +\frac{\partial \psi}{\partial y}$ & $-\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$.

$$\boxed{\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}} \quad \boxed{\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}}.$$

Q6) The velocity potential function ($\phi$) is given by an expression

$$\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3 y}{3} + y^2$$

(i) Find the velocity components in x-direction

(ii) Show that $\phi$ represents a possible case of flow.

Soln Given $\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3 y}{3} + y$.

partial derivatives of $\phi$ w.r.t x & y are

$$\frac{\partial \phi}{\partial x} = \frac{-y^3}{3} - 2x + \frac{3x^2}{3} + 0 \quad \text{—— (I)}$$

$$\frac{\partial \phi}{\partial y} = -\frac{3xy^2}{3} + \frac{x^3}{3} + 2y \quad \text{—— (II)}$$

(i) The velocity components u and v are given by equations

$$u = -\frac{\partial \phi}{\partial x} = -\left[-\frac{y^3}{3} - 2x + \frac{3x^2 y}{3}\right] = \frac{y^3}{3} + 2x - x^2 y$$

$$\boxed{u = \frac{y^3}{3} + 2x - x^2 y}$$ Ans.

& $$v = -\frac{\partial \phi}{\partial y} = -\left[-\frac{3xy^2}{3} + \frac{x^3}{3} + 2y\right]$$

$$\boxed{v = xy^2 - \frac{x^3}{3} - 2y.}$$

(ii) The given value of $\phi$, will represent a possible case of flow if it satisfy the laplace equation i.e

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

from equation (1) & (2) we have

Now
$$\frac{\partial \phi}{\partial x} = -\frac{y^3}{3} - 2x + x^2y.$$

$$\frac{\partial^2 \phi}{\partial x^2} = 0 - 2 + 2xy.$$

or
$$\frac{\partial^2 \phi}{\partial x^2} = 2xy - 2 \quad \text{--- (iii)}$$

&
$$\frac{\partial \phi}{\partial y} = -xy^2 + \frac{x^3}{3} + 2y.$$

$$\frac{\partial^2 \phi}{\partial y^2} = -2xy + 0 + 2.$$

or
$$\frac{\partial^2 \phi}{\partial y^2} = -2xy + 2 \quad \text{--- (iv)}$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (2xy - 2) + (-2xy + 2)$$
$$= \cancel{2xy} - \cancel{2} - \cancel{2xy} + \cancel{2} = 0$$

$\therefore$ Laplace equation is satisfied.
$\phi$ represents a possible case of flow.

H.W page - 185 (Bansal)

Problem (5.11) (5.12), (5.13), (5.14) & (5.15)