564. Repeat Prob. 563 using 2-in. by 10-in. pieces.

5-7 DERIVATION OF FORMULA FOR HORIZONTAL SHEARING STRESS

Consider two adjacent sections, (1) and (2), in a beam separated by the distance dx, as shown in Fig. 5-21, and let the shaded part between them be isolated as a free body. Figure 5-22 is a pictorial representation of this part, the beam from which it is taken being shown in dashed outline.

Assume the bending moment at section (2) to be larger than that at section (1), thus causing larger flexural stresses on section (2) than on section (1). Therefore the resultant horizontal thrust H_2 caused by the compressive forces on section (2) will be greater than the resultant horizontal thrust H_1 on section (1). This difference between H_2 and H_1 can be balanced only by the resisting shear force dF acting on the bottom face of the free body, since no external force acts on the top or side faces of the free body.

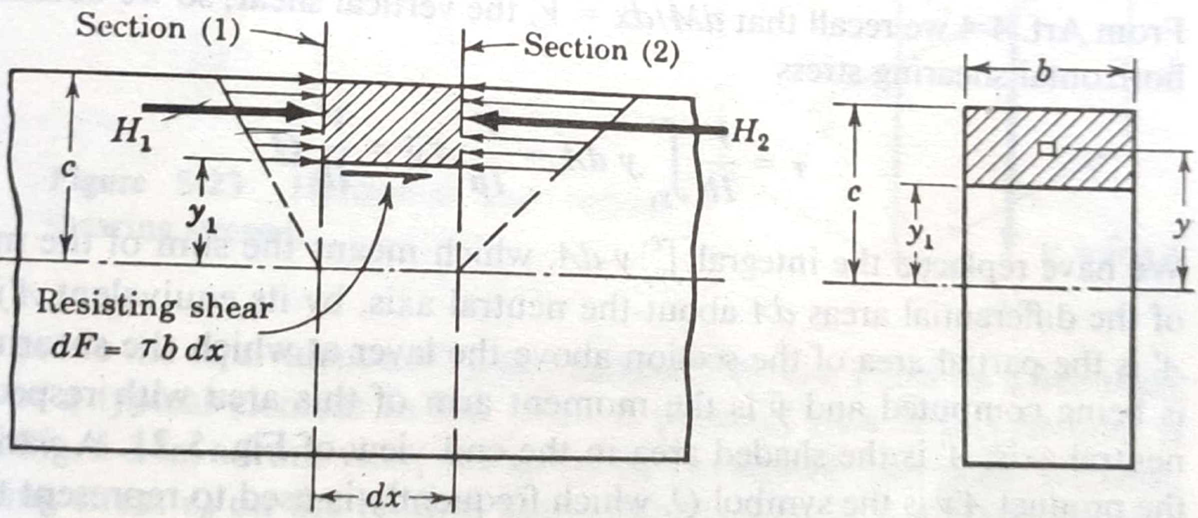


Figure 5-21

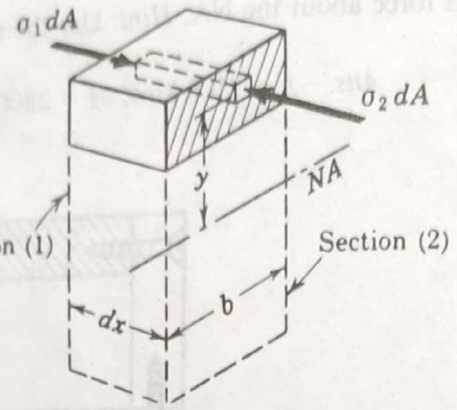


Figure 5-22

Since $H_2 - H_1$ is the summation of the differences in thrusts $\sigma_2\, dA$ and $\sigma_1\, dA$ on the ends of all elements contained in the part shown in Fig. 5-22, a horizontal summation of forces gives

$$[\Sigma F_H = 0] \qquad dF = H_2 - H_1$$

$$= \int_{y_1}^{c} \sigma_2\, dA - \int_{y_1}^{c} \sigma_1\, dA$$

Replacing the flexural stress σ by its equivalent My/I, we obtain

$$dF = \frac{M_2}{I}\int_{y_1}^{c} y\, dA - \frac{M_1}{I}\int_{y_1}^{c} y\, dA = \frac{M_2 - M_1}{I}\int_{y_1}^{c} y\, dA$$

From Fig. 5-21 we note that $dF = \tau b\, dx$, where τ is the average shearing stress over the differential area of width b and length dx; also that $M_2 - M_1$ represents the differential change in bending moment dM in the distance dx; hence the preceding relation is rewritten as

$$\tau = \frac{dM}{Ib\, dx}\int_{y_1}^{c} y\, dA$$

From Art. 4-4 we recall that $dM/dx = V$, the vertical shear; so we obtain for the horizontal shearing stress

$$\tau = \frac{V}{Ib}\int_{y_1}^{c} y\, dA = \frac{V}{Ib} A'\bar{y} = \frac{V}{Ib} Q \qquad (5\text{-}4)$$

We have replaced the integral $\int_{y_1}^{c} y\, dA$, which means the sum of the moments of the differential areas dA about the neutral axis, by its equivalent $A'\bar{y}$, where A' is the partial area of the section above the layer at which the shearing stress is being computed and $\bar{y}$ is the moment arm of this area with respect to the neutral axis; A' is the shaded area in the end view of Fig. 5-21. A variation of the product $A'\bar{y}$ is the symbol Q, which frequently is used to represent the static moment of area.

Shear Flow

If the shearing stress τ is multiplied by the width b, we obtain a quantity q, known as shear flow, which represents the longitudinal force per unit length transmitted across the section at the level y_1. It is analogous to the shear flow discussed previously in the torsion of thin-walled tubes (see page 81). Using Eq. (5-4), we find that its value is given by

$$q = \tau b = \frac{V}{I} Q \tag{5-4a}$$

One application of this relation is discussed in Art. 5-9; another is given in Illustrative Problem 1321 (page 469).

Relation Between Horizontal and Vertical Shearing Stresses

Most students are surprised to find the term *vertical shear* (V) appearing in the formula for horizontal shearing stress (τ_h). However, as we shall show presently, a horizontal shearing stress is always accompanied by an equal vertical shearing stress. It is this vertical shearing stress τ_v, shown in Fig. 5-23, that forms the resisting vertical shear $V_r = \int \tau \, dA$ which balances the vertical shear V. Since it is not feasible to determine τ_v directly, we have resorted to deriving the numerically equal value of τ_h.

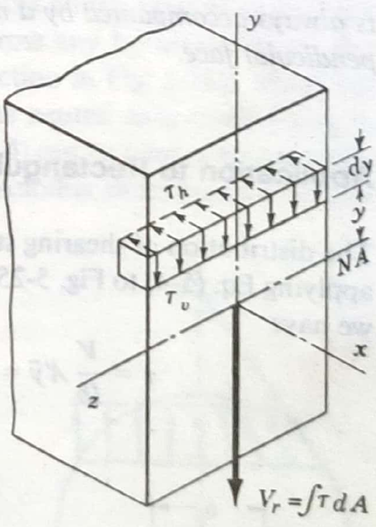
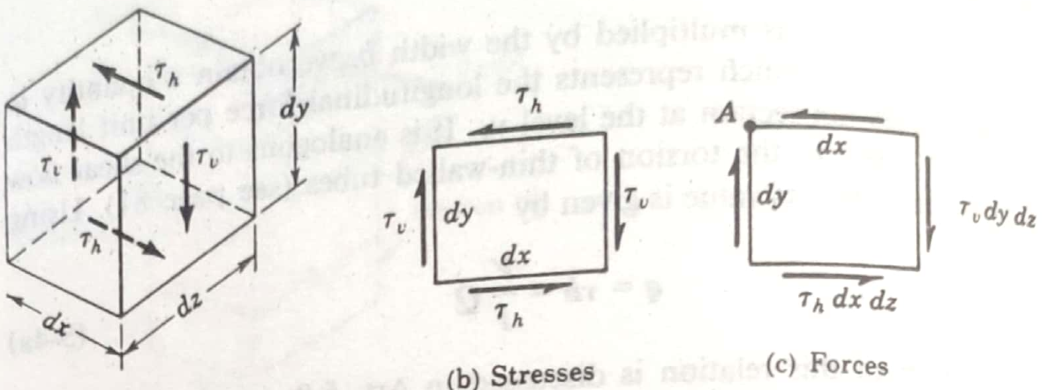


Figure 5-23 Horizontal and vertical shearing stresses.

To prove the equivalence of τ_h and τ_v, consider their effect on a free-body diagram of a typical element in Fig. 5-23. A pictorial view of this element is shown in Fig. 5-24a; a front view, in Fig. 5-24b. For equilibrium of this element, the shearing stress τ_h on the bottom face requires an equal balancing shearing stress on the top face. The forces causing these shearing stresses (Fig. 5-23c) form



(a) (b) Stresses (c) Forces

Figure 5-24 Shearing stresses on a typical element.

a counterclockwise couple, which requires a clockwise couple to ensure balance. The forces of this clockwise couple induce the shearing stresses τ_v on the vertical faces of the element as shown.

By taking moments about an axis through A (Fig. 5-24c), we obtain

$$[\Sigma M_A = 0] \qquad (\tau_h\, dx\, dz)dy - (\tau_v\, dy\, dz)dx = 0$$

from which the constant product $dx\, dy\, dz$ is canceled to yield

$$\tau_h = \tau_v \tag{5-5}$$

We conclude therefore that a shearing stress acting on one face of an element is always accompanied by a numerically equal shearing stress acting on a perpendicular face.

Application to Rectangular Section

The distribution of shearing stresses in a rectangular section can be obtained by applying Eq. (5-4) to Fig. 5-25. For a layer at a distance y from the neutral axis, we have

$$\tau = \frac{V}{Ib} A'\bar{y} = \frac{V}{Ib}\left[b\left(\frac{h}{2} - y\right)\right]\left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right]$$

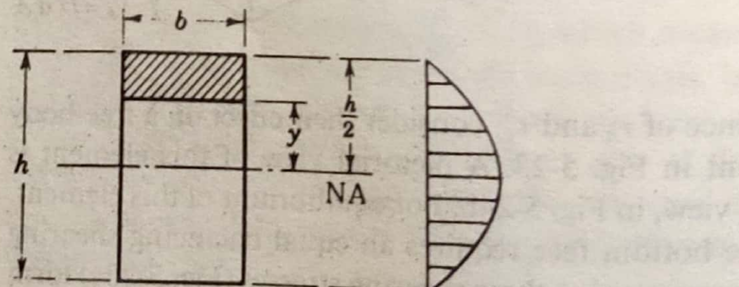


Figure 5-25 Shearing stress is distributed parabolically across a rectangular section.

which reduces to

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)$$

This shows that the shearing stress is distributed parabolically across the depth of the section.

The maximum shearing stress occurs at the neutral axis and is found by substituting the dimensions of the rectangle in Eq. (5-4), as follows:

$$\tau = \frac{V}{Ib}A'\bar{y} = \frac{V}{(bh^3/12)b}\left(\frac{bh}{2}\right)\left(\frac{h}{4}\right)$$

which reduces to

$$\text{Max. } \tau = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A} \tag{5-6}$$

This indicates that the maximum shearing stress in a rectangular section is 50% greater than the average shear stress.

Assumptions and Limitations of Formula

We have assumed, without saying so implicitly, that the shearing stress is uniform across the width of the cross section. Although this assumption does not hold rigorously, it is sufficiently accurate for sections in which the flexure forces are evenly distributed over a horizontal layer.

This condition is present in a rectangular section and in the wide-flange section shown in Fig. 5-26a, where the flexure forces on the vertical strips, both shaded and unshaded, are evenly distributed across any horizontal layer. But this condition does not exist in the triangular section in Fig. 5-26b, where the shearing stress is maximum at the left edge of the neutral axis, diminishing to zero at the right edge. Even here, however, Eq. (5-4) can be used to compute the *average* value of shearing stress across any layer. Another exception is a circular

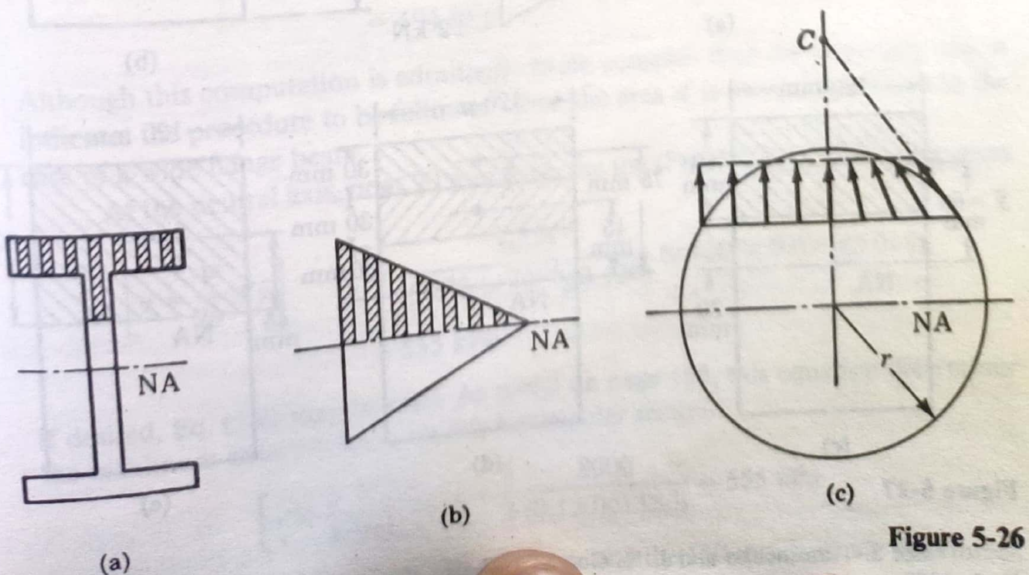


Figure 5-26

cross section (Fig. 5-26c). It can be shown that the stress at the edge of any layer must be tangent to the surface, as in the right half of the figure; but the direction of shearing stresses at interior points is unknown, although they are assumed to pass through a common center C as shown. The vertical components of these shearing stresses are usually assumed to be uniform across any layer, as in the left half of the figure, and are computed by means of Eq. (5-4). With this assumption, the maximum shearing stress across the neutral axis is $\frac{4}{3}(P/\pi r^2)$. A more elaborate study* shows that shearing stress actually varies at the neutral axis from $1.23P/\pi r^2$ at the edges to $1.38P/\pi r^2$ at the center.

ILLUSTRATIVE PROBLEMS

565. A simply supported beam 120 mm wide, 180 mm deep, and 6 m long carries a uniformly distributed load of 4 kN/m, as shown in Fig. 5-27. (a) Compute the shearing stress developed at horizontal layers 30 mm apart from top to bottom for a section 1.0 m from the left end. (b) Compute the maximum shearing stress developed in the beam.

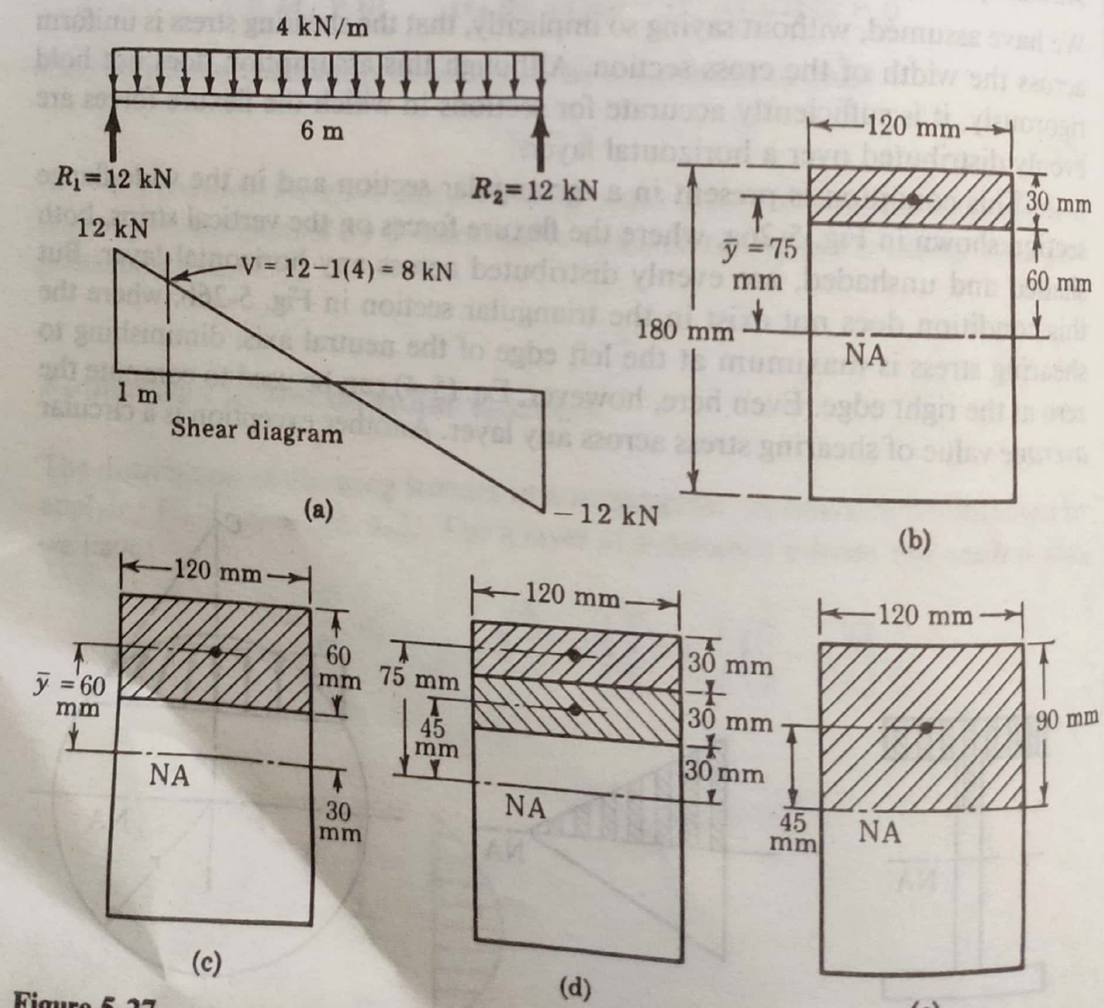


Figure 5-27

* See S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1970.

Solution:

Part a. As shown on the shear diagram (Fig. 5-27a), the definition of vertical shear $V = (\Sigma F_y)_L$ gives $V = 8$ kN at $x = 1.0$ m.

The moment of inertia about the neutral axis is

$$\left[I = \frac{bh^3}{12}\right] \qquad I_{NA} = \frac{120(180)^3}{12} = 58.32 \times 10^6 \text{ mm}^4$$

$$= 58.32 \times 10^{-6} \text{ m}^4$$

Applying Eq. (5-4) to a layer 30 mm from the top (Fig. 5-27b), we find that the shearing stress is

$$\left[\tau = \frac{V}{Ib} A'\bar{y}\right] \qquad \tau = \frac{8000}{(58.32 \times 10^{-6})(0.120)}(0.120 \times 0.030)(0.075)$$

$$= 309 \text{ kPa}$$

Note that for consistency of units, the shear force is expressed in newtons (N), the moment of inertia in quartic meters (m^4) and the distances in meters (m).

At 60 mm from the top (Fig. 5-27c), the shearing stress is

$$\left[\tau = \frac{V}{Ib} A'\bar{y}\right] \qquad \tau = \frac{8000}{(58.32 \times 10^{-6})(0.120)}(0.120 \times 0.060)(0.060)$$

$$= 494 \text{ kPa}$$

The shearing stress at 60 mm from the top can also be computed from Fig. 5-27d, in which the area A' is resolved into two strips 30 mm thick. Since a moment of area equals the sum of the moments of area of its parts (i.e., $A'\bar{y} = \Sigma ay$), an identical result is obtained as follows:

$$\left[\tau = \frac{V}{Ib} \Sigma ay\right] \qquad \tau = \frac{8000}{(58.32 \times 10^{-6})(0.120)}[(0.120 \times 0.030)(0.075)$$

$$+ (0.120 \times 0.030)(0.045)]$$

$$= 494 \text{ kPa}$$

Although this computation is admittedly more complex than the preceding one, it indicates the procedure to be followed when the area A' is more complex, as in the case of a wide-flange beam.

At the neutral axis, or at 90 mm from the top (Fig. 5-27e), the shearing stress is

$$\left[\tau = \frac{V}{Ib} A'\bar{y}\right] \qquad \tau = \frac{8000}{(58.32 \times 10^{-6})(0.120)}(0.120 \times 0.090)(0.045)$$

$$= 555 \text{ kPa}$$

If desired, Eq. (5-6) may be used. As noted on page 165, this equation determines the maximum shearing stress on any rectangular section.

$$\left[\tau = \frac{3}{2}\frac{V}{bh}\right] \qquad \tau = \frac{3}{2}\frac{8000}{(0.120)(0.180)} = 555 \text{ kPa}$$

The shearing stress at the 120-mm layer and the 150-mm layer are determined similarly to be 494 and 309 kPa, respectively.

Note that equal values of τ are obtained at layers equidistant from the NA in any beam symmetrical about the neutral axis. Physically, this is true because, as was said on page 158, the compressive and tensile flexure forces between these layers cancel each other. Analytically, it is true because the neutral axis is the centroidal axis, and hence the moment of area $A'\bar{y}$ computed for a partial area A' located above the NA equals that for a symmetrically placed area below the NA. Further, since the total moment of area is zero with respect to a centroidal axis, it follows that the moment of area about the NA of the area above any layer equals that of the area below that layer. Stated differently, in computing $A'\bar{y}$ we may use either the area above or that below any layer, depending on which is easier to use.

Part b. The maximum shearing stress occurs at the NA of the section of maximum shear. The shear diagram shows that maximum shear occurs at either end, and hence from Eq. (5-6) the maximum shearing stress is

$$\left[\tau = \frac{3}{2}\frac{V}{A}\right] \qquad \text{Max. } \tau = \frac{3}{2}\frac{12 \times 10^3}{(0.120 \times 0.180)} = 833 \text{ kPa} \qquad \textit{Ans.}$$

566. A wide-flange beam has the section shown in Fig. 5-28a. At a cross section where the vertical shear is $V = 16$ kips, compute (a) the maximum shearing stress and (b) the shearing stress at the junction of the flange and the web. (c) Plot the shearing stress distribution in the web and determine the percentage of shear carried by the web alone.

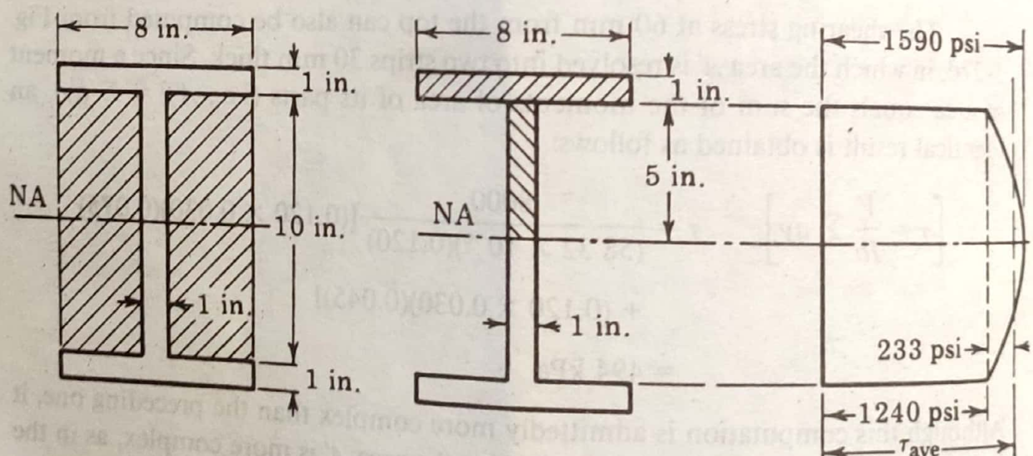


Figure 5-28

Solution: The moment of inertia is found by resolving the section into a large rectangle from which is subtracted the two shaded rectangles. We obtain

$$\left[I = \Sigma I = \Sigma \frac{bh^3}{12}\right] \qquad I_{NA} = \frac{8(12)^3}{12} - 2\left[\frac{3.5(10)^3}{12}\right] = 568 \text{ in.}^4$$

The maximum shearing stress occurs at the neutral axis. In applying Eq. (5-4), compute $A'\bar{y}$ as the sum of the moments of area of the rectangles shaded in Fig. 5-28b.

$$\left[\tau = \frac{V}{Ib}A'\bar{y}\right] \qquad \text{Max. } \tau = \frac{16\,000}{568(1)}[(8 \times 1)(5.5) + (5 \times 1)(2.5)] = 1590 \text{ psi}$$

At the junction of the web and flange, the shearing stress is

$$\left[\tau = \frac{V}{Ib}A'\bar{y}\right] \qquad \tau = \frac{16\,000}{568(1)}(8 \times 1)(5.5) = 1240 \text{ psi}$$

These stresses vary parabolically from top to bottom of the web, as shown in Fig. 5-28c. The average height of the parabolic segment is $\frac{2}{3}$ of (1590 − 1240), or 233 psi. The average shear stress in the web therefore is

$$\tau_{ave.} = 1240 + 233 = 1473 \text{ psi}$$

The shearing force in the web is

$$[P = A\tau_{ave.}] \qquad V_{web} = (10 \times 1)(1473) = 14\,730 \text{ lb}$$

The percentage of shear carried by the web alone is

$$\%\ V_{web} = \frac{14\,730}{16\,000} \times 100 = 92.2\%$$

This shows that the flanges are almost ineffective in resisting the vertical shear. If it is assumed that the total vertical shear is carried by the web alone, the average shearing stress in the web will be very close to the maximum shearing stress as computed from Eq. (5-4). Thus

$$\left[\tau = \frac{V}{A_{web}}\right] \qquad \tau = \frac{16\,000}{10(1)} = 1600 \text{ psi}$$

This is very close to the computed maximum, 1590 psi.

This method gives results that closely approximate the actual maximum τ. In most design specifications or codes, however, the height of the web is not taken as the distance between flanges, but is assumed to be the total depth of the beam. This procedure is not so accurate as the previous method, but lower allowable shearing stresses are usually specified in order to compensate.

PROBLEMS

567. A timber beam 80 mm wide by 160 mm high is subjected to a vertical shear $V =$ 40 kN. Determine the shearing stress developed at layers 20 mm apart from top to bottom of the section.

568. Show that the shearing stress developed at the neutral axis of a beam with circular cross section is $\tau = \frac{4}{3}(V/\pi r^2)$. Assume that the shearing stress is uniformly distributed across the neutral axis.

569. Show that the maximum shearing stress in a beam having a thin-walled tubular section of net area A is $\tau = 2V/A$.

570. A uniformly distributed load of 200 lb/ft is carried on a simply supported span. If the cross section is as shown in Fig. P-570, determine the maximum length of the beam if the shearing stress is limited to 80 psi. Assume the load acts over the entire length of the beam.

Ans. 12.6 ft

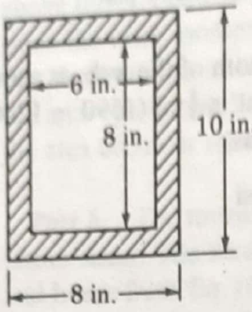


Figure P-570 and P-571

571. For a beam with the same cross section as that in Prob. 570, plot the shearing stress distribution across the cross section at a section where the shearing force is $V = 1800$ lb.

572. The T section shown in Fig. P-572 is the cross section of a beam formed by joining two rectangular pieces of wood together. The beam is subjected to a maximum shearing force of 60 kN. Show that the NA is 34 mm from the top and that $I_{NA} = 10.57 \times 10^6\ \text{mm}^4$. Using these values, determine the shearing stress (a) at the neutral axis and (b) at the junction between the two pieces of wood.

Ans. (a) 3.28 MPa; (b) 3.18 MPa, 31.8 MPa

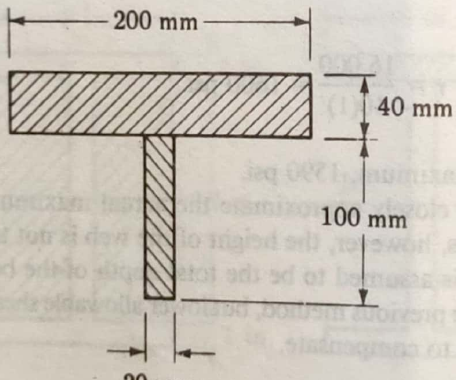


Figure P-572

573. The cross section of a beam is an isosceles triangle with vertex uppermost, of altitude h and base b. If V is the vertical shear, show that the maximum shearing stress is $3V/bh$ located at the midpoint of the altitude.

574. In the beam section shown in Fig. P-574, prove that the maximum horizontal shearing stress occurs at a layer $h/8$ above or below the NA.

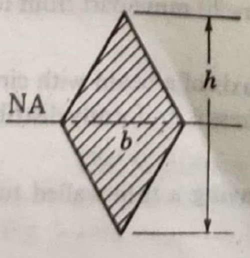


Figure P-574