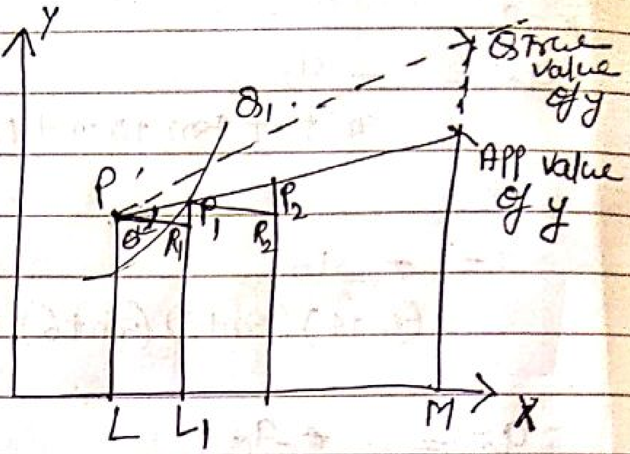
1 **Euler's method** EE+ECE (2nd Sem) Module-3

Solution of ordinary differential equation (By Numerical method)

Consider the first order differential equation

$$\frac{dy}{dx} = f(x,y) \quad \text{—(1)}$$



given that $y(x_0) = y_0$ is the curve solution through $P(x_0, y_0)$ / Similarly

$y(x_1) = y_1$

$y(x_n) = y_n$

thus $P(x_0,y_0)$, $P_1(x_1,y_1)$ ---- $P(x_n,y_n)$ respective solutions Now we have to find the y-axis distance (ordinate) of any other point Q on this curve

Now let divide LM into n-sub interval each of width h at P tangent meets ordinates from L_1 at $P_1(x_0+h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$y_1 = y_0 + h f(x_0, y_0) \quad \text{by (1)}$$

Similarly $y_2 = y_1 + hf(x_1, y_1)$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

Note:- first order differential equation solved by this method but not practical (slow method)

Thus approximate solution of (1) is

$$y(x_n) = y_n$$

$$\boxed{\text{where } y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})}$$

(2)

Example on Euler's method

Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y=1$ at $x=0$

find y for $x = 0.1$

Soln! → ~~dix~~ given that $y(x_0) = y_0$

$\Rightarrow y(0) = 1$

find ~~that~~that $y(x_n) = y_n$

$\Rightarrow y(0.1) = y_n = ?$

divide the interval $[0, 0.1]$ into 5 subintervals.

x	y	$dy/dx = \frac{y-x}{y+x}$	$y_n = y_{n-1} + h \times \left(\frac{dy}{dx}\right)_{(x_{n-1}, y_{n-1})}$
$0.00 = x_0$	$1.0000 = y_0$	$\frac{1.00 - 0.00}{1.00 + 0.00} = 1$	$y_1 = 1.00 + (0.02) \times 1 = 1.0200$
$0.02 = x_1$	$1.020 = y_1$	$\frac{1.020 - 0.02}{1.020 + 0.02} = 0.9615$	$y_2 = 1.020 + (0.02) \times 0.9615 = 1.0392$
0.04	$1.0392 = y_2$	$\frac{1.0392 - 0.04}{1.0392 + 0.04} = 0.926$	$y_3 = 1.0392 + (0.02) \times 0.926 = 1.0577$
0.06	$1.0577 = y_3$	0.893	$1.0577 + (0.02) \times 0.893 = 1.0756 = y_4$
0.08	$1.0756 = y_4$	0.862	$1.0756 + (0.02) \times 0.862 = 1.0928$
0.10	$1.0928 = y_5$		

Hence the required approximate value of $y = 1.0928$

$\boxed{\text{Assignment 1}}$ — Using Euler's method, find approximate value of y when $x = 0.6$ of $\frac{dy}{dx} = 1 - 2xy$, given that $y=0$ when $x=0$ (take $h=0.2$)

$\boxed{\text{Ans} - 0.4748}$

$\boxed{\text{Modified Euler's method}} \Rightarrow \boxed{\text{Runge - Kutta method}}$ $\boxed{\text{Second order}}$

By formula

$$y_1 = y_0 + h\left[f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}hf(x_0, y_0)\right\}\right]$$

$$\boxed{y_{n+1} = y_n + h\left[f\left\{x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}hf(x_n, y_n)\right\}\right]}$$

(3)

Ex→ Solve the equation $\frac{dy}{dx} = 1-y$, given $y(0)=0$ using Modified Euler's method and tabulate the solution at $x=0.1$, 0.2 and 0.3 compare your results with the exact solution.

Soln:- here $x_0 = 0, y_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$

Now by formula

$$y_{n+1} = y_n + hf\left[x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}h f(x_n, y_n)\right]$$

$$\cancel{x_1 = 0 + hf\left[x_0 + \frac{1}{2}(0.1),\ 0 + \frac{1}{2}(0.1)[1-0]\right]}$$

$$y_1 = y_0 + hf\left[x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}h f(x_0, y_0)\right]$$

$$= 0 + 0.1\, f\left[0 + \frac{0.1}{2},\ 0 + \frac{1}{2}(0.1)\left\{\frac{dy}{dx}_{(x_0,y_0)}\right\}\right]$$

$$= (0.1) f\left[\frac{0.1}{2},\ \frac{0.1}{2} \times (1-y_0)\right]$$

$$= (0.1) f\left[\frac{0.1}{2},\ \frac{0.1}{2}(1-0)\right]$$

$$= (0.1) f\left(\frac{0.1}{2}, \frac{0.1}{2}\right) = (0.1) \times \left(1 - \frac{0.1}{2}\right)$$

$$= \cancel{0.1 \times 0.9} = 0.095$$

$$y_2 = y_1 + hf\left[x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}h f(x_1, y_1)\right]$$

$$y_2 = 0.095 + (0.1) f\left[0.1 + \frac{0.1}{2},\ 0.095 + \frac{1}{2} \times 0.1\, f(0.1, 0.095)\right]$$

$$= 0.095 + (0.1) f\left[\frac{0.3}{2},\ 0.095 + \frac{0.1}{2}\left(\underbrace{1 - 0.095}_{1-y_1}\right)\right]$$

$$y_2 = 0.18098$$

$$y_3 = y_2 + hf\left[x_2 + \frac{1}{2}h,\ y_2 + \frac{h}{2} f(x_2, y_2)\right]$$

$$= 0.18098 + \frac{0.1}{2} f\left[0.2 + \frac{0.1}{2},\ 0.18098 + \frac{0.1}{2} f(0.2, 0.18098)\right]$$

$$y_3 = \cancel{0.22193}\ 0.258787$$

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(5) Now compare with exact solution is

$\rightarrow$ solution

$$\frac{dy}{dx} = 1-y \Rightarrow \frac{dy}{1-y} = dx$$

$$\Rightarrow -\log(1-y) = x + C$$

$$\log(1-y) = -(x+C)$$

$$\Rightarrow 1-y = e^{-(x+C)} = e^{-x} \times e^{C}$$

$$\Rightarrow 1-y = e^{-x} \times (2.7)^{C}$$

$$\Rightarrow 1-y = e^{-x} A$$

~~y(0.1) =~~

~~at (x_1, y_1)~~

Now at (x_0, y_0)

$(0, 0)$

$$1 - 0 = 1 \cdot A$$

$$\Rightarrow A = 1$$

~~$y(0.1) = e^{-0.1} \times 1$~~

~~$y(0.2) = e^{-0.2}$~~

thus $y = 1 - e^{-x}$

$$y(0.1) = 1 - e^{-0.1} = 0.09516258$$

$$y(0.2) = 1 - e^{-0.2} = 0.181269247$$

$$y(0.3) = 1 - e^{-0.3} = 0.259181779$$

Assignment Using Modified Euler method, get $y(0.2)$, $y(0.4)$, $y(0.6)$ given $\frac{dy}{dx} = y - x^2$, $y(0) = 1$