~~Residues of a complex number~~

Residues of analytic function

Def^n – Residues – If $f(z)$ be analytic at a singular point $z=a$, then by Laurent's Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

The co-efficient of $(z-a)^{-1}$ i.e. b_1 is called the residues of $f(z)$ at $z=a$ and is denoted by $\text{Res}_{z=a} f(z)$

i.e. $$\boxed{\text{Res}_{z=a} f(z) = \frac{1}{2\pi i} \int_c f(z)\,dz}$$

$$\Rightarrow \boxed{\int_c f(z)\,dz = 2\pi i \left[\text{Res}_{z=a} f(z)\right]}$$

where c is the closed curve around the point $z=a$.

Working rule for finding residues

ⓐ **Residue at simple pole**

(i) If $f(z)$ has a simple pole at $z=a$

then
$$\text{Res}\, f(a) = \lim_{z\to a} (z-a)\, f(z)$$

(ii) If $f(z)$ is of the form, at $z=a$

$$f(z) = \frac{\phi(z)}{\psi(z)}, \quad \text{where } \psi(a)=0 \text{ \& } \phi(a) \neq 0$$

$\therefore$ Residue at $z=a$ $= \dfrac{\phi(a)}{\psi'(a)}$

ⓑ **Residue at a pole of order n.**

If $f(z)$ has a pole of order n at $z=a$

then,
$$\text{Res.}(\text{at } z=a) = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \cdot f(z)\right] \right\}_{z=a}$$

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(2) (c) Residue at a pole $z=a$ of any order (simple or order of m)

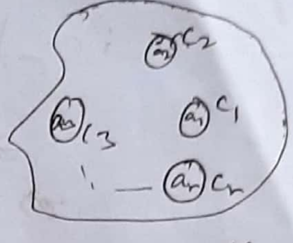
Res. $f(a)$ = coeff. of $(1/t)$

Putting $z=a+t$ in the function $f(z)$, expand it in powers of t

coefficient of $\frac{1}{t}$ is residue of $f(z)$ at $z=a$

Residue theorem

If $f(z)$ is analytic in a closed curve c, with finite number of poles within c, then



$\int_c f(z)dz = 2\pi i$ (sum of residues at the poles within c)

$$\Rightarrow \boxed{\int_c f(z)dz = 2\pi i\left[\text{Res } f(a_1) + \text{Res } f(a_2) + \text{Res. } f(a_3) + \text{———} \text{Res. } f(a_n)\right]}$$

—x—

F.R. Cauchy integral formula is a particular case of this theorem.

Here $[f(z)/z-a]$ has only a simple pole at $z=a$, and the residue is $f(a)$

∴ By residue theorem

$$\boxed{\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)}$$

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Problems on residues

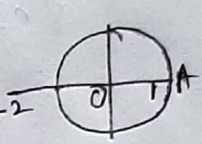
(vii) Find the residue of the following, (1) $\oint_C \frac{3z+5}{z^2+2z} dz$, $C: |z| = 1$

Here $f(z) = \oint_C \frac{3z+5}{z^2+2z} dz$

$= \oint_C \frac{3z+5}{z(z+2)} dz$

Poles are given by $z = 0, z = -2$

i.e. $f(z)$ has a simple poles at $z = 0$, $z = -2$, out of which only $z = 0$ lies within C.



$\therefore \oint_C \frac{3z+5}{z(z+2)} dz = \frac{1}{2} \left[\oint_C (3z+5) \left[\frac{1}{z} - \frac{1}{z+2} \right] dz \right.$ (using partial fraction)

$= \frac{1}{2} \oint_C \frac{3z+5}{z} dz - \frac{1}{2} \oint_C \frac{3z+5}{z+2} dz$

Now, residue at $z = -2$ lies out side the curve,

$\therefore \oint_C \frac{3z+5}{2(z+2)} dz = 0$

& $\frac{1}{2} \oint_C \frac{3z+5}{z} dz$

Residue of $f(z)$ at $z = 0$

$= \frac{1}{2} \lim_{z \to 0} (z-0) f(z)$

$= \frac{1}{2} \lim_{z \to 0} \cancel{z} \cdot \left(\frac{3z+5}{\cancel{z}} \right)$

$= \frac{1}{2} [3 \times 0 + 5] = 5/2$

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(4) $\therefore \oint_C \frac{3z+5}{z}\,dz = 2\pi i \times \text{Res. } f(z) \text{ at } z=0$

$= 2\pi i \times \frac{5}{2} = 5\pi i$

(ii) $\oint_C \frac{\sin z}{z\cos z}\,dz, \quad C: |z| = 2$

Here poles are given by

$z = 0, \pm \pi/2, \pm 3\pi/2, \ldots$

only the poles $z=0$ & $z = \pm\pi/2$ lies inside $|z|=2$

$$\text{Res } f(0) = \lim_{z\to 0}\left[z \cdot f(z)\right]$$

$$= \lim_{z\to 0} \cancel{z} \cdot \frac{\sin z}{\cancel{z}\cos z} = 0$$

$$\text{Res. } f(\pi/2) = \lim_{z\to\pi/2}\left[(z-\pi/2)\, f(z)\right]$$

$$= \lim_{z\to\pi/2}\left[\frac{(z-\pi/2)\sin z}{z\cos z}\right], \text{ form } (0/0)$$

$$= \lim_{z\to\pi/2}\left[\frac{(z-\pi/2)\cos z + \sin z}{\cos z - z\sin z}\right]$$

(using L Hospital's rule)

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

$$\text{Res } f(-\pi/2) = \lim_{z\to -\pi/2}\left[\frac{(z+\pi/2)\sin z}{z\cos z}\right], \text{ form } (0/0)$$

$$= \lim_{z\to -\pi/2}\left[\frac{(z+\pi/2)\cos z + \sin z}{\cos z - z\sin z}\right] = \frac{2}{\pi}$$

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(5) (5)

$$\therefore \oint_C \frac{\sin z}{z\cos z}\,dz = 2\pi i\,[\text{Sum of residues}]$$
$$= 2\pi i\,[\text{Res } f(0) + \text{Res } f(\pi/2) + \text{Res. } f(-\pi/2)]$$
$$= 2\pi i\left[0 - \frac{2}{\pi} + \frac{2}{\pi}\right] = 0$$

(iii) $\displaystyle\oint_C \frac{e^z - 1}{z(z-1)(z-i)^2}\,dz$, $C: |z| = 1/2$

Here $f(z)$ has simple poles at $z = 0$ & $z = 1$

& pole of order 2 at $z = i$

only $z = 0$ lies within C

$$\therefore \text{Res } f(0) = \lim_{z\to 0} (z-0)\, f(z)$$
$$= \lim_{z\to 0} \cancel{z}\cdot\frac{e^z - 1}{\cancel{z}(z-1)(z-i)^2}$$
$$= \frac{e^0 - 1}{(0-1)(0-i)^2}$$
$$= \frac{1-1}{-i^2} = 0$$

$$\therefore \oint_C \frac{e^z - 1}{z(z-1)(z-i)^2}\,dz = 2\pi i\,[\text{Res } f(0)] = 2\pi i \times 0 = 0$$

(iv) $\displaystyle\oint_C \frac{z-3}{z^2+2z+5}\,dz$, (a) $C: |z| = 1$, (b) $C: |z+1-i| = 2$

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⑥

Sol^n: -

Here the poles are given by:

$$z^2+2z+5=0$$

$$z=\frac{-2\pm\sqrt{4-4.1.5}}{2.1}$$

$$=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm 4i}{2}$$

$$=-1\pm 2i$$

(a) Both poles $z=-1+2i$ & $z=-1-2i$

lies outside the circle $|z|=1$

$\therefore$ By Cauchy theorem

$$\oint_C \frac{z-3}{z^2+2z+5}\,dz=0$$

(b) only $z=-1+2i$ lies inside the circle, $|z+1-i|=2$

$\therefore$ $f(z)$ is analytic within C except this pole.

Res $f[-1+2i]$

$$=\lim_{z\to -1+2i}\frac{\cancel{(z+1-2i)}(z-3)}{\cancel{(z+1-2i)}(z+1+2i)}$$

$$=\frac{-1+2i-3}{-\cancel{1}+2i+\cancel{1}+2i}=\frac{-4+2i}{4i}$$

$$=\frac{\cancel{2}(-2+i)}{\cancel{4}_2 i}=-\frac{\cancel{2}}{i}+\frac{\cancel{i}}{2}$$

$$\therefore\ \oint_C \frac{z-3}{z^2+2z+5}\,dz=2\pi i\,[\text{Resi } f(-1+2i)]=2\pi i\,[i+\tfrac{1}{2}]$$

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(7)

(4)

(i) $\oint_C \frac{z^3}{(z-1)^4(z-2)(z-3)}\,dz$, $C: |z| = 2.5$

Sol^n:- Here poles are given by
$z=1$, is of order 4
& $z=2$, $z=3$ are simple poles

Res. f(1)

$$= \frac{1}{(4-1)!}\left[\frac{d^{4-1}}{dz^{4-1}}\left[(z-1)^4 \cdot f(z)\right]\right]_{z=1}$$

$$= \frac{1}{3!}\left[\frac{d^3}{dz^3}\left\{\cancel{(z-1)^4} \times \frac{z^3}{\cancel{(z-1)^4}(z-2)(z-3)}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^3}{dz^3}\left(\frac{z^3}{(z-2)(z-3)}\right)\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^2}{dz^2}\left\{\frac{d}{dz}\left(\frac{z^3}{(z-2)(z-3)}\right)\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^2}{dz^2}\left\{\frac{(z-2)(z-3)3z^2 - z^3 \times 1 \times (z-2) - z^3 \times 1 \times (z-3)}{\{(z-2)(z-3)\}^2}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^2}{dz^2}\left\{\frac{3z^4 - 15z^3 + 18z^2 - z^3(z-2+z-3)}{\{(z-2)(z-3)\}^2}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^2}{dz^2}\left(\frac{z^4 - 10z^3 + 18z^2}{\{(z-2)(z-3)\}^2}\right)\right]_{z=1}$$

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8

$$= \frac{1}{6}\left[\frac{d}{dz}\left(\frac{d}{dz}\left(\frac{z^4 - 10z^3 + 18z^2}{\{(z-2)(z-3)\}^2}\right)\right]_{z=1}$$

$$= \frac{101}{16} \quad \text{(After solving)}$$

Residue $f(2)$

$$= \lim_{z \to 2}\left[(z-2)\frac{z^3}{(z-1)^4 \cancel{(z-2)}(z-3)}\right]_{z=2}$$

$$= \lim_{z \to 2}\left[\frac{z^3}{(z-1)^4 (z-3)}\right]_{z=2}$$

$$= \frac{8}{1 \times (2-3)} = -8$$

Pole $z = 3$ out-side c

$$\therefore \oint_c f(z)\, dz = \oint_c \frac{z^3}{(z-1)^4 (z-2)(z-3)} dz$$

$$= 2\pi i \left[\text{Res } f(1) + \text{Res } f(2)\right]$$

$$= 2\pi i \left[\frac{101}{16} - 8\right]$$

$$= 2\pi i \left[\frac{101 - 128}{16}\right]$$

$$= 2\pi i \left[\frac{-27}{16}\right] = -\frac{27\pi i}{8}$$

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(vi) $\oint_C \frac{e^z}{z^2(z+1)^3} dz$, $C: |z| = 2$

Solⁿ:- Here, $f(z)$ has pole of order 2, at $z=0$
& pole of order 3, at $z=1$

$$\text{Res } f(0) = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left(z^2 \cdot f(z)\right)\right]$$

$$= 1\left[\frac{d}{dz}\left(\cancel{z^2} \times \frac{e^z}{\cancel{z^2}(z+1)^3}\right)\right]_{z=0}$$

$$= \left[\frac{(z+1)^3 \times e^z - e^z \times 3(z+1)^2}{(z+1)^6}\right]_{z=0}$$

$$= \frac{e^0 - e^0 \times 3}{1} = \frac{1-3}{1}$$

$$= -2$$

$$\text{Res. of } f(-1) = \frac{1}{(3-1)!}\left[\frac{d^{3-1}}{dz^{3-1}}\left\{\cancel{(z+1)^3} \times \frac{e^z}{z^2\cancel{(z+1)^3}}\right\}\right]_{z=-1}$$

$$= \frac{1}{2!}\left[\frac{d^2}{dz^2}\left(\frac{e^z}{z^2}\right)\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{d}{dz}\left\{\frac{e^z}{z^2}\right\}\right)\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{z^2 \times e^z - e^z \times 2z}{z^4}\right)\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{\cancel{z} \cdot e^z (z-2)}{\cancel{z^4}\, z^3}\right)\right]_{z=-1}$$

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$$= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{e^z(z-2)}{z^3}\right)\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{z^3\{e^z(z-2)+e^z\times 1\} - e^z(z-2)\times 3z^2}{z^6}\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{z^3\{e^z(z-2+1)\} - e^z(z-2)\times 3z^2}{z^6}\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{z^3\{e^z(z-1)\} - e^z(z-2)\times 3z^2}{z^6}\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{e^z\times \cancel{z^2}\{z(z-1) - 3(z-2)\}}{\cancel{z^6}\, z^4}\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{e^z(z^2-z-3z+6)}{z^4}\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{e^z(z^2-4z+6)}{z^4}\right]_{z=-1}$$

$$= \frac{5}{2}e^{-1}$$

$$\therefore \oint_c \frac{e^z}{z^2(z+1)^3}\,dz = 2\pi i\,[\mathrm{Res}\, f(0) + \mathrm{Res}\, f(-1)]$$

$$= 2\pi i\left[-2 + \frac{5}{2}e^{-1}\right]$$

$$= \pi i\left[-4 + 5e^{-1}\right]$$

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