(A) (7)

(VII) $I = \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$

$= \int_0^{2a}\left[\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dy\right]dx$

$= \int_0^{2a}\left[x^2y+\frac{y^3}{3}\right]_0^{\sqrt{2ax-x^2}}dx$

$= \int_0^{2a} x^2.(2ax-x^2)^{1/2}+\frac{(2ax-x^2)^{3/2}}{3}\,dx$

$= \int_0^{2a}(2ax-x^2)^{1/2}\left[x^2+\frac{1}{3}(2ax-x^2)\right]dx$

$= \int_0^{2a}(2ax-x^2)^{1/2}\left[\frac{2ax+2x^2}{3}\right]dx$

$= \frac{2}{3}\int_0^{2a}(2ax-x^2)^{1/2}(x^2+ax)\,dx$

$= \frac{2}{3}\int_0^{2a} x^{1/2}\cdot\sqrt{2a-x}\cdot x(x+a)\,dx$

$= \frac{2}{3}\int_0^{2a}\sqrt{2a-x}\cdot x^{3/2}(x+a)\,dx$

put $x = 2a\sin^2\theta$

$dx = 4a\sin\theta\cdot\cos\theta\,d\theta$

If $x\to 0,\ \theta\to 0$ & $x\to 2a;\ \theta\to \pi/2$

$= \frac{2}{3}\int_0^{\pi/2}(2a)^{3/2}\cdot\sin^3\theta\cdot a\,(2\sin^2\theta-1)\cdot(2a)^{1/2}\cos\theta\cdot 4a\sin\theta\cos\theta\,d\theta$

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(8) $= \frac{32}{3} a^4 \int_0^{\pi/2} \sin^4\theta \cdot \cos^2\theta \, (2\sin^2\theta + 1) d\theta$

$= \frac{32}{3} a^4 \left[\int_0^{\pi/2} 2\sin^6\theta \cdot \cos^2\theta \, d\theta + \int_0^{\pi/2} \sin^4\theta \cdot \cos^2\theta \, d\theta \right]$

$= \frac{32}{3} a^4 \left[\cancel{2} \cdot \frac{5 \cdot \cancel{3} \cdot 1 \cdot 1}{8 \cdot \cancel{6} \cdot 4 \cdot \cancel{2}} \pi/2 + \frac{\cancel{3} \cdot 1 \cdot 1}{\cancel{6} \cdot 4 \cdot 2} \cdot \pi/2 \right]$

$= \frac{\cancel{32}^{16}}{3} a^4 \left[\frac{5}{64} + \frac{1}{16} \right] \cdot \pi/\cancel{2}$

$= \frac{16\pi a^4}{3} \left[\frac{5+4}{64} \right]$

$= \frac{16\pi a^4}{3} \times \frac{9}{64} = \frac{3\pi a^4}{4}$

(VIII) $I = \int_0^5 \int_0^{x^2} x(x^2+y^2) \, dx\, dy$

$= \int_0^5 \left[\int_0^{x^2} (x^3 + xy^2) \, dy \right] dx$

$= \int_0^5 \left(x^3 y + \frac{xy^3}{3} \right)_0^{x^2} dx$

$= \int_0^5 \left(x^3 \cdot x^2 + \frac{x \cdot x^6}{3} \right) dx$

$= \int_0^5 \left(x^5 + \frac{x^7}{3} \right) dx$

$= \left[\frac{x^6}{6} + \frac{x^8}{24} \right]_0^5 = 5^6 \left(\frac{1}{6} + \frac{25}{24} \right)$

$= 18880.2$ (approx)

(IX) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dx\,dy}{(1+e^y)\sqrt{a^2-x^2-y^2}} =$

(X) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dx\,dy}{(1+e^y)\sqrt{a^2-x^2-y^2}} =$

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4(a) (9)

(1×) $I=\int_0^a\int_0^{\sqrt{a^2-y^2}} \frac{dx\,dy}{(1+e^y)\sqrt{a^2-x^2-y^2}}$

$=\int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \frac{dx}{(1+e^y)\sqrt{a^2-x^2-y^2}}\right] dy$

$=\int_0^a \frac{1}{1+e^y}\left[\int_0^{\sqrt{a^2-y^2}} \frac{dx}{\sqrt{(a^2-y^2)-x^2}}\right] dy$

$=\int_0^a \frac{1}{1+e^y}\left[\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$ $\qquad \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

$=\int_0^a \frac{1}{1+e^y}\left[\sin^{-1}\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right] dy$

$=\int_0^a \frac{1}{1+e^y}\sin^{-1}1\, dy$

$=\int_0^a \frac{1}{1+e^y}\times \pi/2\, dy$

$=\pi/2\int_0^a \frac{1}{1+e^y}\,dy$

$=\pi/2\left[\int_0^a\frac{dt}{t-1}-\int_0^a\frac{dt}{t}\right]$ $\qquad 1+e^y=t$, $e^y dy = dt$, $dy=\frac{dt}{e^y}=\frac{dt}{t-1}$

$=\pi/2\left[\log(t-1)-\log t\right]_0^a$ $\qquad \therefore \int_0^a\frac{dt}{t(t-1)}$

$=\pi/2\left[\log(a-1)-\log a - \log(0-1)+\log 0\right]$ $\qquad \frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}$

$=\pi/2\left[\log\frac{(a-1)}{a}-\log(-1)+1\right]$ $\qquad 1=A(t-1)+Bt$

Put $t=1$,
$1=B$
$t=0$
$1=A(-1)$
$A=-1$
$\therefore \frac{1}{t(t-1)}=\frac{1}{t-1}-\frac{1}{t}$

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(10)

Double integral of polar Co-ordinate

(5)

The function $f(r,\theta)$ be a function of polar Co-ordinates, r,θ is to be integrated.

$$\iint_R f(r,\theta)\,dA = \iint_R f(r,\theta)\,dr\,d\theta$$

$$= \lim_{\substack{n\to\infty \\ \delta A_k \to 0}} \sum_{r=1}^{n} f(r_k,\theta_k)\,\delta A_k \quad -(1)$$

(1) may be expressed as

$$\iint_R f(r,\theta)\,d\theta\,dr = \iint_R f(r,\theta)\,dr\,d\theta$$

$$= \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr.d\theta$$

over the region R bounded by the Curve $r=r_1$, $r=r_2$ & $\theta=\theta_1$, $\theta=\theta_2$ (st. line)

If θ_1 & θ_2 are constants First integrate w.r.to r then integrate w.r.to θ.

Change variable Cartesian (x,y) form to polar Co-ordinate (r,θ)

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

$$= \iint_R f(r,\theta).\,r\,dr\,d\theta$$

(By putting $x = r\cos\theta$
$y = r\sin\theta$
$\delta A = r.\delta r.\delta\theta$
$x^2+y^2 = r^2 \Rightarrow r=\sqrt{x^2+y^2}$
& $\theta = \tan^{-1}(y/x)$

Problems on polar form of double integral

(11)

2. Evaluate (i) $\int_0^{\pi/2}\int_0^{a\cos\theta} r(a^2-r^2)^{1/2}\,dr\,d\theta$

$$= \int_0^{\pi/2}\left[\int_0^{a\cos\theta} r(a^2-r^2)^{1/2}dr\right]d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{a\cos\theta}\left(-\frac{1}{2}\right)(a^2-r^2)^{1/2}(-2r)\,dr\right]d\theta$$

$$= \int_0^{\pi/2}\left[\left(-\frac{1}{2}\right)\cdot\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta}d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\,d\theta$$

$$= -\frac{a^3}{3}\left[\int_0^{\pi/2}(\sin^3\theta-1)\,d\theta\right]$$

$$= -\frac{a^3}{3}\left[\int_0^{\pi/2}\sin^3\theta\,d\theta-\int_0^{\pi/2}d\theta\right]$$

$$= -\frac{a^3}{3}\left[\frac{2}{3}-\pi/2\right]$$

$$= \frac{a^3}{18}(3\pi-4)$$

$$\left[\because \int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\frac{p+1}{2}\cdot\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}\right.$$

$$\int_0^{\pi/2}\sin^3\theta\,d\theta = \frac{\Gamma\frac{3+1}{2}\cdot\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{3+0+2}{2}}$$

$$\left.= \frac{\Gamma 2\cdot\Gamma\frac{1}{2}}{2\cdot\Gamma\frac{5}{2}}\right]$$

$\Gamma\frac{1}{2}=\sqrt{\pi}$

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