For B.Tech -2nd sem — ME/LEATHER TECHNOLOGY

MODULE - I

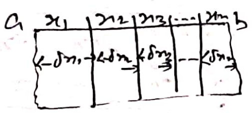
MULTIVARIABLE CALCULUS (INTEGRATION)

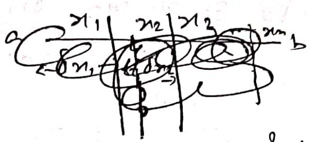
MULTIPLE INTEGRATION.

Double integral (Cartesian form) :- The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \text{———} + f(x_n)\delta x_n$$

if $n \to \infty$ & each of lengths $\delta x_1, \delta x_2 \text{—} \delta x_n \to 0$





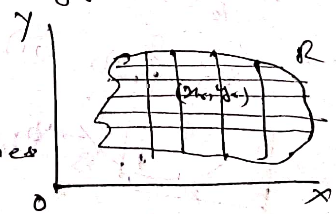
Here, $\delta x_1, \delta x_2$ —— δx_n are n subintervals into the range b-a has been divided

& $x_1, x_2, \text{——} x_n$ are values of x, lies in the 1st, 2nd, —— nth subintervals

A double integral is the counter part of two dimensions.

Let $f(x,y)$ be single valued and bounded function be defined in a closed region R of the xy plane.

Divide the region R into subregions lines drawing ∥ to the co-ordinate axes



Number of rectangles which lie entirely inside the region R from 1 to n.

Let (x_r, y_r) be any point inside the rth rectangle, whose area δA_r may be expressed as a sum

$$\boxed{f(x_1,y_1)\delta A_1 + f(x_2,y_2)\delta A_2 + \text{———} + f(x_n,y_n)\delta A_n = \sum_{r=1}^{n} f(x_r,y_r)\,\delta A_r}$$ ①

(2)

(1) may be expressed as

$$\boxed{\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \lim_{\substack{n\to\infty \\ \delta A\to 0}} \sum_{r=1}^{n} f(x_r,y_r)\,\delta A_r}$$

or

$$= \iint_R f(x,y)\,dy\,dx$$

Evaluation of Double integrals

If the region R be bounded by the curves $x=x_1$, $x=x_2$ and $y=y_1$, $y=y_2$, then there are three types of Double integrals.

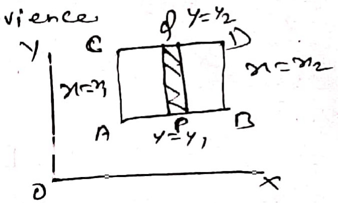
(i) If x_1, x_2, y_1, y_2 are constants :–

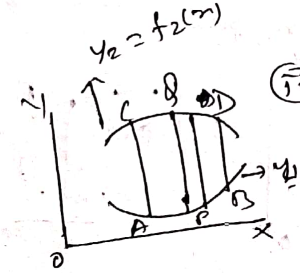
$$I = \iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy$$

or

$$\int_{x_1}^{x_2}\boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx$$

i.e. either integrate w.r.t x or y as per convience

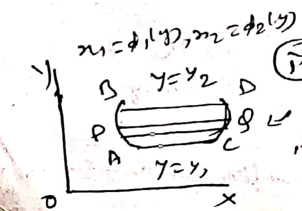




(ii) If y_1, y_2 are functions of x & x_1, x_2 are constants then

$$I = \iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx$$

i.e. Integrate first w.r.t y then integrate w.r.t x



(iii) If x_1, x_2 are functions of y and y_1, y_2 are constants

$$\therefore I = \iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy$$

i.e integrate 1st w.r.t x then integrate w.r.t y

Problems on Double integrals

(1) (i) Evaluate $\int_1^2\int_1^3 xy^2\,dx\,dy$

$$I = \int_1^2\int_1^3 xy^2\,dx\,dy = \int_1^2 x\left(\int_1^3 y^2\,dy\right)dx$$

$$= \int_1^2 x\left(\frac{y^3}{3}\right)_1^3 dx = \int_1^2 x\left(\frac{27}{3}-\frac{1}{3}\right)dx$$

$$= \frac{26}{3}\int_1^2 x\,dx = \frac{26}{3}\left[\frac{x^2}{2}\right]_1^2 = \frac{26}{3}\times\left(\frac{4}{2}-\frac{1}{2}\right)$$

$$= \frac{26}{3}\times\frac{3}{2}$$

$= 13$ sq. units

(ii) $I = \int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dx\,dy$

$$= \int_0^1\left[\int_x^{\sqrt{x}}(x^2+y^2)\,dy\right]dx$$

$$= \int_0^1\left(x^2y+\frac{y^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_0^1\left\{x^2(\sqrt{x}-x)+\frac{(\sqrt{x})^3}{3}-\frac{x^3}{3}\right\}dx$$

$$= \int_0^1\left(x^{5/2}-x^3+\frac{x^{3/2}}{3}-\frac{x^3}{3}\right)dx$$

$$= \left[\frac{x^{7/2}}{7/2}-\frac{x^4}{4}+\frac{1}{3}\frac{x^{5/2}}{5/2}-\frac{1}{3}\frac{x^4}{4}\right]_0^1$$

$$= \left[\frac{2}{7}x^{7/2}+\frac{2}{15}x^{5/2}-\frac{x^4}{4}\left(1+\frac{1}{3}\right)\right]_0^1$$

$$= \left[\frac{2}{7}x^{7/2}+\frac{2}{15}x^{5/2}-\frac{1}{4}\cdot\frac{4}{3}x^4\right]_0^1$$

$$= \frac{2}{7}(1-0)+\frac{2}{15}(1-0)-\frac{1}{3}(1-0)$$

$$= \frac{2}{7}+\frac{2}{15}-\frac{1}{3} = \frac{3}{35}$$

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(iii) $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

$= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx \right] dy$

$= \int_0^a \left[\frac{x \cdot \sqrt{a^2-y^2-x^2}}{2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$

$\left[\because \int \sqrt{a^2-x^2}\, dx = x \cdot \frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$

$= \int_0^a \left(0 + \frac{a^2-y^2}{2} \cdot \sin^{-1} \frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}} \right) dy$

$= \int_0^a \frac{a^2-y^2}{2} \cdot \sin^{-1} 1\, dy$

$= \pi/4 \int_0^a (a^2-y^2)\, dy$

$= \pi/4 \left[a^2 y - \frac{y^3}{3} \right]_0^a = \pi/4 \left[a^3 - \frac{a^3}{3} \right]$

$= \frac{\pi}{4_2} \times \frac{\cancel{2}a^3}{3} = \frac{\pi a^3}{6}$

(iv) $I = \int_0^1 \int_0^y xy\, e^{-x^2}\, dy\, dx$

$= \int_0^1 y \left[\int_0^y x \cdot e^{-x^2} dx \right] dy$

$= \int_0^1 \left(-\frac{y}{2}\right) \left[\int_0^y (-2x)\, e^{-x^2} dx \right] dy$

$= \int_0^1 \left(-\frac{y}{2}\right) \left[\int_0^y d(e^{-x^2}) \right] dy$

$$= \int_0^1 \left(-\frac{y}{2}\right) \left(e^{-x^2}\right)_0^y dy$$

$$= \int_0^1 (-y/2)\left(e^{-y^2} - e^0\right) dy$$

$$= \int_0^1 (-y/2)\left(e^{-y^2} - 1\right) dy$$

$$= \frac{1}{2}\left[\int_0^1 -y e^{-y^2} dy + \int_0^1 y\, dy\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\int_0^1 d(e^{-y^2}) + \frac{1}{2}\left(y^2\right)_0^1\right]$$

$$= \frac{1}{4}\left[e^{-y^2} + y^2\right]_0^1 = \frac{1}{4}\left[e^{-1} + 1 - 1 - 0\right]$$

$$= \frac{1}{4} \cdot \frac{1}{e} = \frac{1}{4e}$$

(v) $$I = \int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$$

$$= \int_0^4 \left[\int_0^{x^2} e^{y/x} dy\right] dx$$

$$= \int_0^4 \left(\frac{e^{y/x}}{1/x}\right)_0^{x^2} dx$$

$$= \int_0^4 \left(x\, e^{y/x}\right)_0^{x^2} dx$$

$$= \int_0^4 \left(x \cdot e^{x} - x \cdot e^0\right) dx$$

$$= \int_0^4 \left(x e^{x} - x\right) dx$$

$$= \int_0^4 x e^{x} dx - \int_0^4 x\, dx$$

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$$= \left[x \cdot \int e^x dx - \int \left(\partial x \int e^x dx\right) dx\right]_0^4 - \left[\frac{x^2}{2}\right]_0^4$$

$$= \left[x \cdot e^x - \int e^x dx\right]_0^4 - \left(\frac{x^2}{2}\right)_0^4$$

$$= \left[(4 \cdot e^4 - e^4) - 0 + 1\right] - \left(\frac{16}{2} - 0\right)$$

$$= 3e^4 + 1 - 8 = 3e^4 - 7$$

(vi) $$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right] dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2 + y^2}\right] dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right)_0^{\sqrt{1+x^2}} dx$$

$$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1} 1 \, dx$$

$$= \pi/4 \int_0^1 \frac{dx}{\sqrt{1+x^2}} dx$$

$$= \pi/4 \left[\log(x + \sqrt{1+x^2})\right]_0^1 \quad \left[\because \int \frac{dx}{\sqrt{a^2+x^2}} = \log(x+\sqrt{a^2+x^2})\right]$$

$$= \pi/4 \left[\log(1+\sqrt{2}) - \log 1\right]$$

$$= \pi/4 \log(\sqrt{2}+1)$$