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Solving ordinary and partial differential Equation by Laplace Transformation

Laplace transformation can be used to solve ordinary and partial differential equation.

We know that:-,

$$L[f'(t)] = s\bar{f}(s) - f(0)$$ (Derivative of Laplace transformation)

In general $L[f^n(t)] = s^n \bar{f}(s) - s^{n-1} f(0)$

$- s^{n-2} f'(0) - \_\_\_\_$

e.g. 1) Solve by the method of transformations the equation

$y''' + 2y'' - y' - 2y = 0$, given $y(0) = y'(0) = 0$ & $y''(0) = 6$

Soln:- Given

$$y''' + 2y'' - y' - 2y = 0 \quad \text{--- (1)}$$

Taking Laplace transformation both sides we get

$$L[y'''] + 2L(y'') - L(y') - 2L(y) = 0$$

$$\Rightarrow [s^3\bar{y} - s^2 y(0) - s y'(0) - y''(0)] + 2[s^2\bar{y} - s\,y(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$

$$\Rightarrow s^3\bar{y} - 0 - 0 - 6 + 2[s^2\bar{y} - 0 - 0] - [s\bar{y} - 0] - 2\bar{y} = 0$$

$$\Rightarrow (s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$\Rightarrow$ ~~$s^3 + 2s^2 - s - 2$~~

(2)

$$\Rightarrow \bar{y} = \frac{6}{(s^3+2s^2-s-2)}$$

$$= \frac{6}{s^2(s+2)-1(s+2)}$$

$$= \frac{6}{(s^2-1)(s+2)}$$

$$= \frac{6}{(s-1)(s+1)(s+2)}$$

$$= 6\left[\frac{1}{6(s-1)} - \frac{1}{2(s+1)} + \frac{1}{3(s+2)}\right]$$

$$\bar{y} = \frac{1}{(s-1)} - \cancel{\frac{6}{3}}\, 3\frac{1}{s+1} + 2\cdot\frac{1}{s+2}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left(\frac{1}{s+1}\right) + 2L^{-1}\left(\frac{1}{s+2}\right)$$

$$\left[\because L^{-1}(\bar{y}) = y\right.$$
i.e. $L[f(t)] = f(s)$
$$\left.\Rightarrow f(t) = L^{-1}(f(s))\right]$$

$$\Rightarrow \boxed{y = e^t - 3e^{-t} + 2e^{-2t}}$$

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e.g (2) Solve $(D^2+4D+4)y = e^{-t}$, given that $y(0) = y'(0) = 0$

Solⁿ:- Given diffⁿ equation may be written as

$$y''+4y'+4y = e^{-t} \quad \text{—(1)}$$

Taking Laplace transformation both sides we get

$$L(y'') + 4L(y') + 4L(y) = L(e^{-t})$$

③

$$\Rightarrow s^2\bar{y} - s\,y(0) - y'(0) + 4[s\bar{y} - y(0)] + 4\bar{y} = \frac{1}{s+1}$$

$$\Rightarrow s^2\bar{y} - 0 - 0 + 4s\bar{y} - 0 + 4\bar{y} = \frac{1}{s+1}$$

$$\Rightarrow (s^2+4s+4)\bar{y} = \frac{1}{s+1}$$

$$\Rightarrow \bar{y} = \frac{1}{(s+1)(s^2+4s+4)} = \frac{1}{(s+1)(s+2)^2}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$\bar{y} = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2} \quad \text{—— (1)}$$

i.e. $1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$

putting $s=-1$, we get

$1 = A(-1+2)^2 \Rightarrow A = 1$

Putting $s=-2$, $C=-1$

Putting $s=0 \Rightarrow B = -1$

$$\therefore \bar{y} = \frac{1}{(s+1)} - \frac{1}{(s+2)} - \frac{1}{(s+2)^2}$$

$$\Rightarrow L^{-1}(\bar{y}) = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$\Rightarrow y = e^{-t} - e^{-2t} - e^{-2t}\cdot L^{-1}(1/s^2)$$

$$= e^{-t} - e^{-2t} - e^{-2t}\,t$$

$$= e^{-t} - (1+t)\,e^{-2t}$$

(4)

e.g. (3) Solve

$$(D^3 - 3D^2 + 3D - 1)\,y = t^2 e^t,$$

given that $y(0) = 1$, $y'(0) = 0$ & $y''(0) = -2$

Sol^n:— Given diff^n equation be

$$y''' - 3y'' + 3y' - y = t^2 e^t \quad \text{——(1)}$$

Taking Laplace transformation both sides we get

$$\left[s^3\bar{y} - s^2 y(0) - s y'(0) - y''(0)\right] - 3\left[s^2\bar{y} - s y(0) - y'(0)\right] + 3\left[s\bar{y} - y(0)\right] - \bar{y} = \frac{2}{(s-1)^3}$$

$$\left[\because L(t^2 e^t) = \frac{2}{(s-1)^3}\right]$$

$$\Rightarrow \left[s^3\bar{y} - s^2 - 0 + 2\right] - 3\left[s^2\bar{y} - s - 0\right] + 3\left[s\bar{y} - 1\right] - \bar{y} = \frac{2}{(s-1)^3}$$

$$\Rightarrow [s^3 - 3s^2 + 3s - 1]\bar{y}$$

$$- (s^2 + 2 - 3s + 3) = \frac{2}{(s-1)^3}$$

$$\Rightarrow (s-1)^3 \bar{y} - (s^2 - 3s + 1] = \frac{2}{(s-1)^3}$$

$$\Rightarrow \bar{y} = \frac{(s^2 - 3s + 1)}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

taking inverse we get

$$\Rightarrow y = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$- L^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$+ 2L^{-1}\left[\frac{1}{(s-1)^6}\right]$$

$$= e^t \left[1 - t - \tfrac{1}{2} t^2 + \frac{1}{60} t^5\right]$$

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e.g. (4) using Laplace transformation solve the differential equation

$$y'' - 3y' - 4y = 2e^{-t} \text{ with } y(0) = 1 = y'(0)$$

(6) Sol^n:-

Given that

$y'' - 3y' - 4y = 2e^{-t}$ —(1)

Taking Laplace transformation both sides

$L(y'') - 3L(y') - 4L(y) = 2L(e^{-t})$

$\Rightarrow s^2\bar{y} - sy(0) - y'(0) - 3[s\bar{y} - y(0)] - 4\bar{y} = \frac{2}{s+1}$

$\Rightarrow s^2\bar{y} - s - 1 - 3s\bar{y} + 3 - 4\bar{y} = \frac{2}{s+1}$

$\Rightarrow (s^2 - 3s - 4)\bar{y} - s + 2 = \frac{2}{(s+1)}$

$\Rightarrow (s^2 - 3s - 4)\bar{y} = \frac{2}{s+1} + s - 2$

$\Rightarrow \bar{y} = \frac{\cancel{2} + s^2 - 2s + s - \cancel{2}}{(s+1)(s^2 - 3s - 4)}$

$\Rightarrow \bar{y} = \frac{s^2 - s}{(s+1)[s^2 - 4s + s - 4]}$

$\Rightarrow \bar{y} = \frac{s(s-1)}{(s+1)[s(s-4) + 1(s-4)]}$

$= \frac{s(s-1)}{(s+1)(s+1)(s-4)} = \frac{s(s-1)}{(s+1)^2(s-4)}$

Now, resolve into partial fraction

$\frac{s(s-1)}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$\Rightarrow s(s-1) = A(s+1)(s-4) + B(s-4) + C(s+1)^2$

put $s=-1$, we get

$2 = -5B$, $B = \cancel{\quad} -\frac{2}{5}$

$s=4$, $12 = 25C \Rightarrow C = \frac{12}{25}$

$s=0$, $0 = -4A - 4B + C$

$= -4A + \frac{8}{5} + \frac{12}{25}$

$\Rightarrow A = \frac{13}{25}$

$\therefore \bar{y} = \frac{s(s-1)}{(s+1)^2(s-4)} = \frac{13}{25} \times \frac{1}{(s+1)} - \frac{2}{5} \times \frac{1}{(s+1)^2} + \frac{12}{25} \times \frac{1}{s-4}$

taking inverse of Laplace we get

$y = \frac{13}{25} \times e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} \cdot e^{4t}$

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e.g. 5 Solve $(D^2+n^2)x = a \sin(nt+\alpha)$,

$x = Dx = 0$, at $t=0$

Sol^n:- Given $(D^2+n^2)x = a \sin(nt+\alpha)$

$\Rightarrow (D^2+n^2)x = a[\sin nt \cdot \cos\alpha + \cos nt \cdot \sin\alpha]$

Taking Laplace transformation both sides, we get

(8) $\Rightarrow [s^2\bar{x} - s x(0) - x'(0)] + n^2\bar{x} = a\cos\alpha\, L(\sin nt)$

$+ a\sin\alpha\, L(\cos nt)$

$\Rightarrow [s^2\bar{x} - s x(0) - x'(0)] + n^2\bar{x}$

$= a\cos\alpha \times \frac{n}{s^2+n^2} + a\sin\alpha \frac{s}{s^2+n^2}$

$\Rightarrow \bar{x} = an\cos\alpha \cdot \frac{1}{(s^2+n^2)^2} + a\sin\alpha \cdot \frac{s}{(s^2+n^2)^2}$

taking inverse of Laplace transformation

$x = an\cos\alpha \cdot \frac{1}{2n^3}[\sin nt - nt\cos nt]$

$+ a\sin\alpha \cdot \frac{t}{2n} \sin nt$

$= \frac{a[\sin nt \cdot \cos\alpha - nt\cos(nt+\alpha)]}{2n^2}$

---x---

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