(1)

Fourier Transforms (Generalise form of Fourier series)

We know that Fourier Series is defined as in periodic f^n in the interval $(-l, l)$

i.e. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + [a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \text{———} + a_n \cos nx + \text{———}]$$

$$+ [b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \text{———}]$$

Where $$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Since $\sin x$ & $\cos x$ are periodic between 0 to 2π.

$\therefore a_0, a_n, b_n$ may also expressed as

(2)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

i.e. In other words A periodic function $f(x)$ defined in a finite interval $(-l, l)$ can be expressed as a Fourier series.

When we define the non periodic function in $-\infty < x < \infty$, $\forall x$ can be expressed as a Fourier integral.

i.e. A function $f(x)$ is piecewise continuous in every finite interval and is absolutely integrable on the x-axis, can be expressed

(3)

by a Fourier integral

$$f(x) = \int_0^\infty [A(\lambda) \cdot \cos\lambda x + B(\lambda)\sin\lambda x]\, d\lambda$$

is valid ~~for~~ at all pts of Continuity.

At a point discontinuity x_0, the Fourier integral is the average of the left and right hand limits

$$= \frac{1}{2}[f(x_0-0) + f(x_0+0)]$$

Standard results

1. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

2. $\int_0^\infty e^{-ax} \cdot \sin bx\, dx = \frac{b}{a^2+b^2}$

3. $\int e^{ax} \cdot \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

4. ~~$\int_0^\infty e^{-ax} \sin bx\, dx =$~~

4. $\int_0^\infty e^{-ax} \cdot \cos bx\, dx = \frac{a}{a^2+b^2}$

Scanned with CamScanner

(4) (5) $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

(6) $\int_0^{\infty} \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}}\,dx = \frac{1}{2}\tan\frac{a}{2}$

(7) $\int_0^{\infty} \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}}\,dx = \frac{1}{2}\sec a/2$

(8) $\int_0^{\infty} \frac{e^{-ax}}{x} \cdot \sin bx\,dx = \tan^{-1} b/c,$
$c > 0, b > 0$

(9) $\int_0^{\infty} \frac{\sin ax}{x}\,dx = \pi/2$ if $a > 0$

Fourier Transform

From the fourier integral representation we get
Fourier ~~integral~~ transform (which is complex).
Fourier cosine & sine transform (which are real) of f(x)

i.e Fourier transform of f(x)

The Fourier integral of f(x) in the complex form is given by

(5) (8)

classmate Date Page

$$f(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-i\alpha t}\, dt \right] e^{i\alpha x}\, d\alpha \quad \text{--- (1)}$$

(1) may be written as

$$f(x) = \int_{-\infty}^{\infty} F(\alpha) \cdot e^{i\alpha x}\, d\alpha \quad \text{--- (2)}$$

$$\text{Where } F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\alpha x}\, dx \quad \text{--- (3)}$$

$F(\alpha)$ defined on (3) is known as the Fourier transform of $f(x)$.

$f(x)$ defined by (2) is called the inverse Fourier transform of $F(\alpha)$.

Here $F(\alpha)$ & $f(x)$ are known as Fourier transform pair which differ in form only in the sign of the exponent.

Scanned with CamScanner

The factor $\frac{1}{2\pi}$ can be multiply the $f(x)$ integral (2) instead of the $F(\alpha)$ integral (3) otherwise the factor $\frac{1}{\sqrt{2\pi}}$ can multiply each of the integral in (2) & (3).

Fourier transform breaks up the function into a continuous spectrum of frequencies α.

Fourier transform method is the process of obtaining $F(\alpha)$ for a given function $f(x)$.