16.1 Introduction

A machine part subjected to an axial compressive force is called a ***strut.*** A strut may be horizontal, inclined or even vertical. But a vertical strut is known as a ***column, pillar*** or ***stanchion.*** The machine members that must be investigated for column action are piston rods, valve push rods, connecting rods, screw jack, side links of toggle jack etc. In this chapter, we shall discuss the design of piston rods, valve push rods and connecting rods.

Note: The design of screw jack and toggle jack is discussed in the next chapter on 'Power screws'.

16.2 Failure of a Column or Strut

It has been observed that when a column or a strut is subjected to a compressive load and the load is gradually increased, a stage will reach when the column will be subjected to ultimate load. Beyond this, the column will fail by crushing and the load will be known as ***crushing load.***

Depending on the end conditions, different columns have different crippling loads

It has also been experienced, that sometimes, a compression member does not fail entirely by crushing, but also by bending *i.e.* buckling. This happens in the case of long columns. It has also been observed, that all the *short columns fail due to their crushing. But, if a **long column is subjected to a compressive load, it is subjected to a compressive stress. If the load is gradually increased, the column will reach a stage, when it will start buckling. The load, at which the column tends to have lateral displacement or tends to buckle is called ***buckling load, critical load,*** or ***crippling load*** and the column is said to have developed an elastic instability. The buckling takes place about the axis having minimum radius of gyration or least moment of inertia. It may be noted that for a long column, the value of buckling load will be less than the crushing load. Moreover, the value of buckling load is low for long columns, and relatively high for short columns.

16.3 Types of End Conditions of Columns

In actual practice, there are a number of end conditions for columns. But we shall study the Euler's column theory on the following four types of end conditions which are important from the subject point of view:

1. Both the ends hinged or pin jointed as shown in Fig. 16.1 (*a*),
2. Both the ends fixed as shown in Fig. 16.1 (*b*),
3. One end is fixed and the other hinged as shown in Fig. 16.1 (*c*), and
4. One end is fixed and the other free as shown in Fig. 16.1 (*d*).

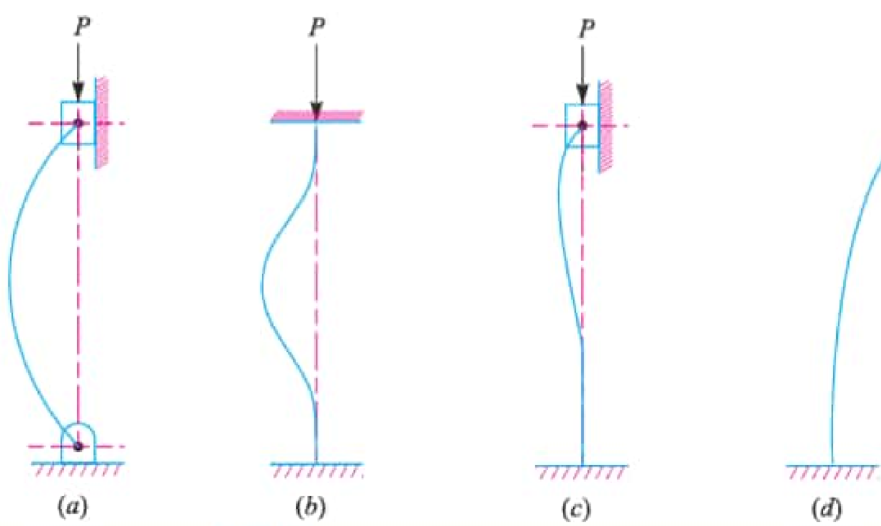

Fig. 16.1. Types of end conditions of columns.

16.4 Euler's Column Theory

The first rational attempt, to study the stability of long columns, was made by Mr. Euler. He derived an equation, for the buckling load of long columns based on the bending stress. While deriving this equation, the effect of direct stress is neglected. This may be justified with the statement, that the direct stress induced in a long column is negligible as compared to the bending stress. It may be noted that Euler's formula cannot be used in the case of short columns, because the direct stress is considerable, and hence cannot be neglected.

16.5 Assumptions in Euler's Column Theory

The following simplifying assumptions are made in Euler's column theory :

1. Initially the column is perfectly straight, and the load applied is truly axial.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic, and thus obeys Hooke's law.
4. The length of column is very large as compared to its cross-sectional dimensions.
5. The shortening of column, due to direct compression (being very small) is neglected.
6. The failure of column occurs due to buckling alone.
7. The weight of the column itself is neglected.

16.6 Euler's Formula

According to Euler's theory, the crippling or buckling load (W_{cr}) under various end conditions is represented by a general equation,

$$W_{cr} = \frac{C \pi^2 E I}{l^2} = \frac{C \pi^2 E A k^2}{l^2} \qquad \ldots (\because I = A.k^2)$$

$$= \frac{C \pi^2 E A}{(l / k)^2}$$

where
E = Modulus of elasticity or Young's modulus for the material of the column,
A = Area of cross-section,
k = Least radius of gyration of the cross-section,
l = Length of the column, and
C = Constant, representing the end conditions of the column or end fixity coefficient.

The following table shows the values of end fixity coefficient (C) for various end conditions.

Table 16.1. Values of end fixity coefficient (C).

S. No.	End conditions	End fixity coefficient (C)
1.	Both ends hinged	1
2.	Both ends fixed	4
3.	One end fixed and other hinged	2
4.	One end fixed and other end free	0.25

Notes : 1. The vertical column will have two moment of inertias (*viz.* I_{xx} and I_{yy}). Since the column will tend to buckle in the direction of least moment of inertia, therefore the least value of the two moment of inertias is to be used in the relation.

2. In the above formula for crippling load, we have not taken into account the direct stresses induced in the material due to the load which increases gradually from zero to the crippling value. As a matter of fact, the combined stresses (due to the direct load and slight bending), reaches its allowable value at a load lower than that required for buckling and therefore this will be the limiting value of the safe load.

16.7 Slenderness Ratio

In Euler's formula, the ratio l/k is known as ***slenderness ratio.*** It may be defined as the ratio of the effective length of the column to the least radius of gyration of the section.

It may be noted that the formula for crippling load, in the previous article is based on the assumption that the slenderness ratio l/k is so large, that the failure of the column occurs only due to bending, the effect of direct stress (*i.e.* W/A) being negligible.

This equipment is used to determine the crippling load for axially loaded long struts.

16.8 Limitations of Euler's Formula

We have discussed in Art. 16.6 that the general equation for the crippling load is

$$W_{cr} = \frac{C\,\pi^2\,E\,A}{(l/k)^2}$$

∴ Crippling stress,

$$\sigma_{cr} = \frac{W_{cr}}{A} = \frac{C\,\pi^2\,E}{(l/k)^2}$$

A little consideration will show that the crippling stress will be high, when the slenderness ratio is small. We know that the crippling stress for a column cannot be more than the crushing stress of the column material. It is thus obvious that the Euler's fromula will give the value of crippling stress of the column (equal to the crushing stress of the column material) corresponding to the slenderness ratio. Now consider a mild steel column. We know that the crushing stress for mild steel is 330 N/mm² and Young's modulus for mild steel is 0.21×10^6 N/mm².

Now equating the crippling stress to the crushing stress, we have

$$\frac{C\,\pi^2\,E}{(l/k)^2} = 330$$

$$\frac{1 \times 9.87 \times 0.21 \times 10^6}{(l/k)^2} = 330 \qquad \text{... (Taking } C = 1)$$

or $$(l/k)^2 = 6281$$

$\therefore$ $$l/k = 79.25 \text{ say } 80$$

Hence if the slenderness ratio is less than 80, Euler's formula for a mild steel column is not valid.

Sometimes, the columns whose slenderness ratio is more than 80, are known as ***long columns,*** and those whose slenderness ratio is less than 80 are known as ***short columns.*** It is thus obvious that the Euler's formula holds good only for long columns.

16.9 Equivalent Length of a Column

Sometimes, the crippling load according to Euler's formula may be written as

$$W_{cr} = \frac{\pi^2 E I}{L^2}$$

where L is the equivalent length or effective length of the column. The equivalent length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends to that of the given column. The relation between the equivalent length and actual length for the given end conditions is shown in the following table.

Table 16.2. Relation between equivalent length (L) and actual length (l).

S.No.	*End Conditions*	*Relation between equivalent length (L) and actual length (l)*
1.	Both ends hinged	$L = l$
2.	Both ends fixed	$L = \frac{l}{2}$
3.	One end fixed and other end hinged	$L = \frac{l}{\sqrt{2}}$
4.	One end fixed and other end free	$L = 2l$

16.10 Rankine's Formula for Columns

We have already discussed that Euler's formula gives correct results only for very long columns. Though this formula is applicable for columns, ranging from very long to short ones, yet it does not give reliable results. Prof. Rankine, after a number of experiments, gave the following empirical formula for columns.

$$\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} \qquad \ldots(i)$$

where W_{cr} = Crippling load by Rankine's formula,

W_C = Ultimate crushing load for the column = $\sigma_c \times A$,

W_E = Crippling load, obtained by Euler's formula = $\frac{\pi^2 E I}{L^2}$

A little consideration will show, that the value of W_C will remain constant irrespective of the fact whether the column is a long one or short one. Moreover, in the case of short columns, the value of W_E will be very high, therefore the value of $1 / W_E$ will be quite negligible as compared to 1t / W_C. It is thus obvious, that the Rankine's formula will give the value of its crippling load (*i.e.* W_{cr}) approximately equal to the ultimate crushing load (*i.e.* W_C). In case of long columns, the value of W_E will be very small, therefore the value of $1 / W_E$ will be quite considerable as compared to $1 / W_C$. It is thus obvious, that the Rankine's formula will give the value of its crippling load (*i.e.* W_{cr}) approximately equal to the crippling load by Euler's formula (*i.e.* W_E). Thus, we see that Rankine's formula gives a fairly correct result for all cases of columns, ranging from short to long columns.

From equation (*i*), we know that

$$\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} = \frac{W_E + W_C}{W_C \times W_E}$$

$$\therefore \quad W_{cr} = \frac{W_C \times W_E}{W_C + W_E} = \frac{W_C}{1 + \frac{W_C}{W_E}}$$

Now substituting the value of W_C and W_E in the above equation, we have

$$W_{cr} = \frac{\sigma_c \times A}{1 + \frac{\sigma_c \times A \times L^2}{\pi^2 E I}} = \frac{\sigma_c \times A}{1 + \frac{\sigma_c}{\pi^2 E} \times \frac{A L^2}{A k^2}} \qquad \ldots (\because I = A.k^2)$$

$$= \frac{\sigma_c \times A}{1 + a\left(\frac{L}{k}\right)^2} = \frac{\text{Crushing load}}{1 + a\left(\frac{L}{k}\right)^2}$$

where σ_c = Crushing stress or yield stress in compression,

A = Cross-sectional area of the column,

a = Rankine's constant = $\frac{\sigma_c}{\pi^2 E}$,

L = Equivalent length of the column, and

k = Least radius of gyration.

The following table gives the values of crushing stress and Rankine's constant for various materials.

Table 16.3. Values of crushing stress (σ_c) and Rankine's constant (a) for various materials.

S.No.	Material	σ_c in MPa	$a = \frac{\sigma_c}{\pi^2 E}$
1.	Wrought iron	250	$\frac{1}{9000}$
2.	Cast iron	550	$\frac{1}{1600}$
3.	Mild steel	320	$\frac{1}{7500}$
4.	Timber	50	$\frac{1}{750}$

16.11 Johnson's Formulae for Columns

Prof. J.B. Johnson proposed the following two formula for short columns.

1. *Straight line formula.* According to straight line formula proposed by Johnson, the critical or crippling load is

$$W_{cr} = A\left[\sigma_y - \frac{2\,\sigma_y}{3\pi}\left(\frac{L}{k}\right)\sqrt{\frac{\sigma_y}{3C \times E}}\right] = A\left[\sigma_y - C_1\left(\frac{L}{k}\right)\right]$$

where A = Cross-sectional area of column,

σ_y = Yield point stress,

$$C_1 = \frac{2\,\sigma_y}{3\pi}\sqrt{\frac{\sigma_y}{3C.E}}$$

= A constant, whose value depends upon the type of material as well as the type of ends, and

$\frac{L}{k}$ = Slenderness ratio.

If the safe stress (W_{cr} / A) is plotted against slenderness ratio (L / k), it works out to be a straight line, so it is known as straight line formula.

2. *Parabolic formula.* Prof. Johnson after proposing the straight line formula found that the results obtained by this formula are very approximate. He then proposed another formula, according to which the critical or crippling load,

$$W_{cr} = A \times \sigma_y\left[1 - \frac{\sigma_y}{4\,C\,\pi^2\,E}\left(\frac{L}{k}\right)^2\right] \text{ with usual notations.}$$

If a curve of safe stress (W_{cr} / A) is plotted against (L / k), it works out to be a parabolic, so it is known as parabolic formula.

Fig. 16.4 shows the relationship of safe stress (W_{cr} / A) and the slenderness ratio (L / k) as given by Johnson's formula and Euler's formula for a column made of mild steel with both ends hinged (*i.e.* $C = 1$), having a yield strength, $\sigma_y = 210$ MPa. We see from the figure that point A (the point of tangency between the Johnson's straight line formula and Euler's formula) describes the use of two formulae. In other words, Johnson's straight line formula may be used when $L / k < 180$ and the Euler's formula is used when $L / k > 180$.

Similarly, the point B (the point of tangency between the Johnson's parabolic formula and Euler's formula) describes the use of two formulae. In other words, Johnson's parabolic formula is used when $L / k < 140$ and the Euler's formula is used when $L / k > 140$.

Note : For short columns made of ductile materials, the Johnson's parabolic formula is used.

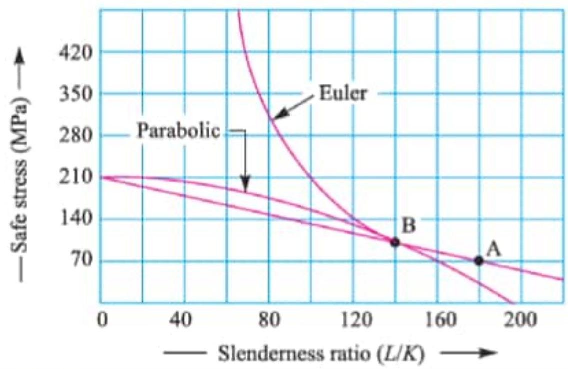

Fig. 16.4. Relation between slenderness ratio and safe stress.

16.12 Long Columns Subjected to Eccentric Loading

In the previous articles, we have discussed the effect of loading on long columns. We have always referred the cases when the load acts axially on the column (*i.e.* the line of action of the load coincides with the axis of the column). But in actual practice it is not always possible to have an axial load on the column, and eccentric loading takes place. Here we shall discuss the effect of eccentric loading on the Rankine's and Euler's formula for long columns.

Consider a long column hinged at both ends and subjected to an eccentric load as shown in Fig. 16.5.

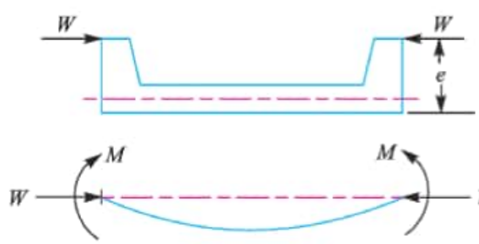

Fig. 16.5. Long column subjected to eccentric loading.

Let W = Load on the column,

A = Area of cross-section,

e = Eccentricity of the load,

Z = Section modulus,

y_c = Distance of the extreme fibre (on compression side) from the axis of the column,

k = Least radius of gyration,

I = Moment of inertia = $A.k^2$,

E = Young's modulus, and

l = Length of the column.

We have already discussed that when a column is subjected to an eccentric load, the maximum intensity of compressive stress is given by the relation

$$\sigma_{max} = \frac{W}{A} + \frac{M}{Z}$$

The maximum bending moment for a column hinged at both ends and with eccentric loading is given by

$$M = W.e.\sec\frac{l}{2}\sqrt{\frac{W}{E.I}} = W.e.\sec\frac{l}{2\,k}\sqrt{\frac{W}{E.A}} \qquad \dots (\because I = A.k^2)$$

$$\therefore \qquad \sigma_{max} = \frac{W}{A} + \frac{W.e.\sec\frac{l}{2\,k}\sqrt{\frac{W}{E.A}}}{Z}$$

$$= \frac{W}{A} + \frac{W.e.y_c.\sec\frac{l}{2\,k}\sqrt{\frac{W}{E.A}}}{A.k^2} \qquad \dots (\because Z = I/y_c = A.k^2/y_c)$$

$$= \frac{W}{A}\left[1 + \frac{e.y_c}{k^2}\sec\frac{l}{2\,k}\sqrt{\frac{W}{E.A}}\right]$$

$$= \frac{W}{A}\left[1 + \frac{e.y_c}{k^2}\sec\frac{L}{2\,k}\sqrt{\frac{W}{E.A}}\right]$$

... (Substituting $l = L$, equivalent length for both ends hinged).