①

First ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

CE Math II

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

How to find partial differential equations

(i) By eliminating arbitrary constants.
(ii) By eliminating arbitrary functions.

Ex→1

~~From~~ the partial diff eqn from

$$z = f(x^2 - y^2) \quad \text{—①}$$

Soln:- ~~we have~~ differentiating ① w.r.t x and y, we get

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)\, 2x \quad \text{—②}$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \quad \text{—③}$$

Now Dividing ② by ③, we get

$$\frac{p}{q} = \frac{-x}{y} \Rightarrow py = -qx$$

$$\Rightarrow py + qx = 0$$

Linear eqn of the first order

which is also known as Lagrange's linear equation of

Equation of the Type $Pp + Qq = R$

where P, Q, R are function of x, y, z.

This eqn is called quasi-linear equation

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(2) when P, Q and R are independent of z it is known as linear equation

by solving auxiliary eqn let the two solution be $u = c_1$ and $v = c_2$

Then $f(u,v) = 0$ or $u = \phi(v)$ is the required solution of $Pp + Qq = R$

Ex-1 Solve

$$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy \quad \text{--- (1)}$$

The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - z^2 - zx + xy} = \frac{dz - dx}{z^2 - x^2 - xy + yz}$$

$$\frac{dx - dy}{(x^2 - y^2) + (zx - yz)} = \frac{dy - dz}{(y^2 - z^2) + (xy - zx)} = \frac{dz - dx}{z^2 - x^2 + yz - xy}$$

$$\Rightarrow \frac{dx - dy}{(x+y)(x-y) + z(x-y)} = \frac{dy - dz}{(y+z)(y-z) + x(y-z)} = \frac{dz - dx}{(z+x)(z-x) + y(z-x)}$$

$$\Rightarrow \frac{dx - dy}{(x+y+z)(x-y)} = \frac{dy - dz}{(x+y+z)(y-z)} = \frac{dz - dx}{(x+y+z)(z-x)}$$

$$\Rightarrow \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} \quad \text{--- (2)}$$

by first two equality

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

$$\Rightarrow \log(x-y) = \log(y-z) + \log c_1$$

$$\Rightarrow \frac{x-y}{y-z} = c_1$$

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(3)

$$\frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

Integrating

$$\log(y-z) = \log(z-x) + \log C_2$$

$$\Rightarrow \frac{y-z}{z-x} = C_2$$

Thus required solution is $f\left[\frac{x-y}{y-z}, \frac{y-z}{z-x}\right] = 0$

~~Method of Multiplif~~

$\boxed{\text{METHOD OF MULTIPLIERS}}$

Let the auxiliary equations be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

l, m, n be constants or functions of x, y, z, then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

l, m, n are chosen in such a way that

$$lP + mQ + nR = 0$$

Thus $l\,dx + m\,dy + n\,dz = 0$

Solve this differential equation, if the solution is $u = C_1$

Similarly, choose another set of multipliers (l_1, m_1, n_1) and if the second solution is v

$v = C_2$

$\therefore$ Required solution $f(u, v) = 0$

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(4) Ex→1

Solve the Partial differential equation

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$$

Where, $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

Soln:- Lagrange's Subsidary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

we see that if we take $x, y, -1$ as multipliers added denominator should be zero

i.e $\frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} \rightarrow$ zero

$\Rightarrow \frac{x\,dx + y\,dy - dz}{0} =$

$\Rightarrow$ $x\,dx + y\,dy - dz = 0$

integrating

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}$$

$$\Rightarrow x^2 + y^2 - 2z = \frac{2c_1}{2}$$

$$\Rightarrow \boxed{\begin{array}{c} x^2+y^2-2z = c_1 \\ f(u) = c \end{array}}$$

Now choose another multipliers for denominator should be zero. Thus

Again, we take $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ as multipliers, we get

each fraction $\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

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(5) $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\therefore$ on integration

$$\cancel{x^2+y^2+z^2=c_1}$$

$$\log x + \log y + \log z = \log c_2$$

$$xyz = c_2$$

Hence the general solution is

$$\phi(x^2+y^2-2z,\ xyz) = 0$$

or $x^2+y^2-2z = f(xyz)$

Assignment

(1) Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2$

(2) $(y+zx)p - (x+yz)q = x^2 - y^2$