(2)

Example on Legendre Polynomial with some more conditions (CE + ME) Math II

Soln:- $P_4(x) =$

Ex-1 Express $f(x) = x^4+3x^3-x^2-5x-2$ in terms of Legendre Polynomial.

Soln:- Let $x^4+3x^3-x^2-5x-2 \equiv aP_4(x)+bP_3(x)+cP_2(x) + d\,P_1(x)+eP_0(x)$ — (1)

We know that $P_n(x)=\sum_{r=0}^{N} \frac{(-1)^r\,(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}$

if n is even thus in $P_n(x)$ put $n-2r=0$, $r=n/2$ in last term [i.e x contain zero power]

and if n is odd thus in $P_n(x)$ put $n-2r=1$, $r=\frac{n-1}{2}$ in last term [i.e x contain one power]

~~thus~~ we know that

$P_0(x)=1$

$P_1(x)=x$

$P_2(x) = \frac{(-1)^0\,(4-0)!}{2^2\,0!\,(2-0)!\,2!}\,x^2 + \frac{(-1)^1\,(2\times2-2)!}{2^2\,1!\,(2-1)!\,(2-2)!}\,x^{2-2}$ → 0 last power

$= \frac{1\cdot 4!\,x^2}{2^2\,2!\,2!} - \frac{2!}{2^2\,1!\,1!}\,x^0$

$= \frac{\cancel{4}\times3\times\cancel{2}}{\cancel{4}\times2\times\cancel{2}}\,x^2 - \frac{1}{2}$

$= \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

Now $P_3(x) = \frac{(-1)^0\,(2\times3-0)!}{2^3\,0!\,(3-0)!\,(3-0)!}\,x^3 + \frac{(-1)^1\,(2\times3-2)!}{2^3\,1!\,(3-1)!\,(3-2)!}\,x^{3-2}$ → last power

$= \frac{\cancel{6}\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}}{\cancel{8}\times\cancel{3}\times\cancel{2}\times\cancel{3}\times2}\,x^3 - \frac{\cancel{4}\times3\times\cancel{2}}{\cancel{8}\times2}\,x = \frac{5}{2}x^3 - \frac{3}{2}x$

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(2) $P_4(x) = \dfrac{(-1)^0 \lfloor 2\times4-0}{2^4 \lfloor 0 \lfloor 4-0 \lfloor 4-0} x^{4-0} + \dfrac{(-1)^1 \lfloor 2\times4-2}{2^4 \lfloor 1 \lfloor 4-1 \lfloor 4-2} x^{4-2}$

$+ \dfrac{(-1)^2 \lfloor \cancel{4\times2} \; 4\times2-2\times2}{2^4 \lfloor 2 \lfloor 4-2 \lfloor 4-4} x^{4-4}$ → last we power

$P_4(x) = \dfrac{8\times7\times6\times5\times4\times3\times2}{16\times4\times3\times2\times4\times3\times2} x^4 - \dfrac{6\times5\times4\times3\times2}{16\times3\times2\times2} x^2 + \dfrac{4\times3\times2}{16\times2\times2}$

$= \dfrac{35x^4}{8} - \dfrac{15x^2}{4} + \dfrac{3}{8}$

above these values

Now put in eq^n (1)

$x^4 + 3x^3 - x^2 - 5x - 2 = a\left(\dfrac{35}{8}x^4 - \dfrac{15}{4}x^2 + \dfrac{3}{8}\right) + b\left(\dfrac{5}{2}x^3 - \dfrac{3}{2}x\right)$
$+ c\left(\dfrac{3}{2}x^2 - \dfrac{1}{2}\right) + dx + e$

Now equating Same power co-efficient

power of x^4 i.e $1 = a\times\dfrac{35}{8} \Rightarrow a = \dfrac{8}{35}$

Power of x^3

$3 = \dfrac{5b}{2} \Rightarrow b = \dfrac{6}{5}$

Power of $x^2 \Rightarrow -\dfrac{15a}{4} + \dfrac{3}{2}c = -1$

$\Rightarrow -\dfrac{15}{4}\times\dfrac{8}{35} + \dfrac{3}{2}c = -1$

$\Rightarrow -\dfrac{30}{35} + \dfrac{3}{2}c = -1$

$\Rightarrow \dfrac{3}{2}c = -1 + \dfrac{30}{35} = \dfrac{-5}{35} = -\dfrac{1}{7}$

$\Rightarrow c = -\dfrac{1}{7}\times\dfrac{2}{3} = \dfrac{-2}{21}$

Power of x

$\Rightarrow -5 = -\dfrac{3b}{2} + d$

$\Rightarrow -5 = -\dfrac{3}{2}\times\dfrac{6}{5} + d$

$\Rightarrow -5 = -\dfrac{9}{5} + d$

$\dfrac{9}{5} - 5 = d$

$\dfrac{9-25}{5} = d$

$\Rightarrow -\dfrac{16}{5} = d$

Power of $x^0 \Rightarrow -2 = \dfrac{3}{8}a - \dfrac{1}{2}c + e \Rightarrow -2 = \dfrac{3}{8}\times\dfrac{8}{35} - \dfrac{1}{2}\times\dfrac{-2}{21} + e$

$\Rightarrow e = \dfrac{-32}{15}$

③

Thus put the value of a, b, c, d, e in ①

$$x^4+3x^3-x^2-5x-2 = \frac{8}{35}P_4(x)+\frac{6}{5}P_3(x)-\frac{2}{21}P_2(x)-\frac{16}{5}P_1(x)$$
$$-\frac{32}{15}P_0(x)$$

Ex-2 Show that

$$x^4 = \frac{8}{5}\cdot\frac{1}{35}\left[8P_4(x)+20P_2(x)+7P_0(x)\right]$$

Soln:-

we have discuss in above example

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_0(x) = 1$

Thus put in R.H.S $\frac{1}{35}\left[8\times\frac{1}{8}\{35x^4-30x^2+3\}\right.$
$\left.+\cancel{20}\times\frac{1}{\cancel{2}}(3x^2-1)+7\times1\right]$

$= \frac{1}{\cancel{35}}\left[\cancel{35}x^4-\cancel{30x^2}+\cancel{3}+\cancel{30x^2}-\cancel{10}+\cancel{7}\right] = x^4$ L.H.S

Generating function for $P_n(x)$

In the expansion of $(1-2xt+t^2)^{-1/2}$, the co-efficient of t^n is known as Generating function for $P_n(x)$

Show that $\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)^{-1/2}$

$\Rightarrow \cancel{t^n\left[\sum_{n=0}^{\infty} P_n(x)\right]} = \{1-\underbrace{t(2x-t)}_{z}\}^{-1/2}$

(4)

We know that by Binomial expansion:

$$(1-z)^{-1/2} = 1+\frac{1}{2}z+\frac{\frac{1}{2}\cdot\frac{3}{2}}{\lfloor 2}z^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{\lfloor 3}z^3+\cdots$$

(n+1)th term

$$= 1+\frac{\lfloor 2}{(\lfloor 1)^2 2^2}z+\cdots\frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\cdots\left(\frac{2n-1}{2}\right)}{\lfloor n}z^n+\cdots$$

$$\therefore \left[1-t(2x-t)\right]^{-1/2} = 1+\frac{\left(-\frac{1}{2}\right)\times -t(2x-t)}{\lfloor 1}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\{-t(2x-t)\}^2}{\lfloor 2}$$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}-2\right)\{-t(2x-t)\}^3}{\lfloor 3}$$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left\{-\frac{1}{2}-(n-1)\right\}\{-t(2x-t)\}^n}{\lfloor n-1}$$

$$+\cdots$$

(n+1)th term

$$= 1+\frac{1}{2}t(2x-t)+\frac{\frac{1}{2}\times\frac{3}{2}}{\lfloor 2}\{t(2x-t)\}^2+\cdots\frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\cdots\left\{\frac{2n-1}{2}\right\}}{\lfloor n-1}\{t(2x-t)\}\cdots$$

$+\cdots$

Now including rth term

$$(r+1)\text{th term} = \frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\cdots\left\{\frac{1}{2}+(r-1)\times 1\right\}\{t(2x-t)\}^r$$

$$= 1+\frac{\lfloor 2}{(\lfloor 1)^2 2^2}t(2x-t)+\frac{\lfloor 4}{(\lfloor 2)^2 2^4}t^2(2x-t)^2+\frac{\lfloor 6}{(\lfloor 3)^2 2^6}t^3(2x-t)^3$$

$$+\cancel{\frac{\lfloor 2n-2r}{[\lfloor n-r]^2 2^{2n-2r}}t^{n-r}(2x-t)^{n-r}}+\frac{\lfloor 2n}{(\lfloor n)^2 2^{2n}}t^n(2x-t)^n$$

~~For getting $t^{n-r}(2x-t)^{n-r}$~~

The term t^n from the term containing $t^{n-r}(2n-r)^{n-r}$

$$\frac{\lfloor 2n-2r}{[\lfloor n-r]^2 2^{2n-2r}}t^{n-r}\,{}^{(n-r)}C_r(-t)^r(2x)^{n-2r}$$

$$= \cancel{(-1)^r\lfloor 2n-2r}\;\frac{\lfloor 2n-2r}{[\lfloor n-r]^2 2^{2n-2r}}(-1)^r t^{r-n}\frac{\lfloor n-r}{\lfloor r\,\lfloor n-2r}2^{n-2r}x^{n-2r}$$

$$= (-1)^r\frac{\lfloor 2n-2r\; x^{n-2r}\times t^n}{2^n\lfloor r\,\lfloor n-r\,\lfloor n-2r}$$

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Thus all the co-efficient term of t^n is $P_n(x)$

where $P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r \, \lfloor 2n-2r}{2^n \lfloor r \, \lfloor n-r \, \lfloor n-2r} x^{n-2r}$

where $\cancel{r} = \frac{n}{2} \to x^0$ when n is even | last term is x^0 or x^1

$\Rightarrow n-2r=0$

$r = \frac{n-1}{2} \to x^1$ when n is odd

$n-2r=1$

Hence we see that

$$\left[1-t(2x-t)\right]^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{—(A)}$$

~~kno~~ which is known as generating function of Legendre Polynomials.

Condition I

~~$P_1(x) = 1$~~

$P_n(1) = 1$

Now put $x=1$ in A

$$\left[1-t(2-t)\right]^{-1/2} = \sum_{n=0}^{\infty} P_n(1) t^n$$

$$\Rightarrow \left[1-2t+t^2\right]^{-1/2} = \sum_{n=0}^{\infty} P_n(1) t^n$$

$$\Rightarrow \left[(1-t)^2\right]^{-1/2} = \sum_{n=0}^{\infty} P_n(1) t^n$$

$$\Rightarrow (1-t)^{-1} = \cancel{1+t+t^2+} \; 1+t+t^2+t^3+\dots+t^n+\dots$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(1) t^n = 1+t+t^2+t^3+\dots+t^n$$

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Thus equating the co-efficient of t^n

$\Rightarrow P_n(1) = 1$

Condition 2

$$P_n(-1) = (-1)^n$$

Taking $x=-1$ in (A)

$$\sum_{n=0}^{\infty} P_n(-1)t^n = (1+t)^{-1} = 1-t+t^2-\dots+(-t)^n$$

equating the co-efficient of t^n is $(-1)^n$

thus $P_n(-1) = (-1)^n$

(7) Show that

$$P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5 --- (2n-1)}{2\cdot 4\cdot 6 --- 2n}$$

$$P_{2n+1}(0) = 0$$

Same as $P_n(0) = \begin{cases} (-1)^{n/2} \dfrac{1\cdot 3\cdot 5 --- (n-1)}{2\cdot 4\cdot 6 --- n}, & \text{when } n \text{ is even} \\ = 0 & \text{when } n \text{ is odd} \end{cases}$

we know that

$$\sum t^{2n} P_{2n}(0) = (1-2xt+t^2)^{-1/2}$$

$$= (1+t^2)^{-1/2}$$

$$= 1 + \left(-\frac{1}{2}\right)t^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\lfloor 2} t^4 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\lfloor 3} t^6 + ---$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{1}{2} - n+1\right)}{\lfloor n} t^{2n} + ---$$

equating the co-efficient of t^{2n} both sides, we get

$$P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5 --- (2n-1)}{2^n \lfloor n}$$

$$= (-1)^n \frac{1\cdot 3\cdot 5 --- (2n-1)}{2\cdot 4\cdot 6 --- 2n}$$

equating the co-efficient of t^{2n+1} both side

but in above expression only even ~~terms~~ power are presenting thus

$$P_{(2n+1)}(0) = 0 \quad \text{Proved//}$$

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(8) Prove that

(i) $P_n'(1) = \frac{1}{2} n(n+1)$

(ii) $P_n'(-1) = (-1)^{n-1} \frac{n}{2}(n+1)$

Soln:- we know that Legendre's equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad -(1)$$

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \quad -(2)$$

$\left[\because P_n(x) \text{ is the solution of (1)}\right.$

thus

~~$(1-x^2)P_n''(1)$~~ put $x=1$ in (2)

$$0 - 2P_n'(1) + n(n+1)P_n(1) = 0$$

$$2P_n'(1) = n(n+1)P_n(1)$$

$$P_n'(1) = \frac{n(n+1)}{2}P_n(1) = \frac{n(n+1)}{2} \quad \left[\because P_n(1) = 1\right.$$

On Putting $x=-1$ in (2)

$$0 + 2P_n'(-1) + n(n+1)P_n(-1) = 0$$

$$2P_n'(-1) + (-1)^n n(n+1)P_n(1) = 0$$

$$\Rightarrow 2P_n'(-1) + (-1)^n n(n+1) = 0$$

$$\Rightarrow 2P_n'(-1) - (-1)^{n-1} n(n+1) = 0$$

$$\Rightarrow P_n'(-1) = (-1)^{n-1}\frac{n(n+1)}{2}$$

$\boxed{\text{Recurrence formulae}}$

(1) $(n+1)P_{(n+1)}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Soln:-

$$(1-2xt+t^2)^{-1/2} = \sum t^n P_n(x)$$

Diff w.r.t t $\quad -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t)$

$$= \sum n t^{n-1} P_n(x)$$

(5)

Multiplying both side by $(1-2xt+t^2)$, we get

$$(1-2xt+t^2)^{-1/2}(x-t) = (1-2xt+t^2)\sum n t^{n-1} P_n(x)$$

$$(x-t)\sum t^n P_n(x) = (1-2xt+t^2)\sum n t^{n-1} P_n(x)$$

~~equating the co-efficient of t^n both side~~

$$\Rightarrow x\sum t^n P_n(x) - t\sum t^n P_n(x)$$

$$= \sum n t^{n-1} P_n(x)$$

$$-2xt\sum n t^{n-1} P_n(x)$$

$$+t^2\sum n t^{n-1} P_n(x)$$

for getting co-efficient of t^n

we should have to adjust in Polynomial function

$\Rightarrow$ Thus co-efficent of t^n in both side is equal

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x)$$

$$-2xn P_n(x)$$

$$+ \cancel{n P_{n-1}} + (n-1) P_{n-1}(x)$$

$$\Rightarrow \boxed{(n+1) P_{n+1}(x) = x P_n(x)\{1+2n\} - n P_{n-1}(x)}$$

(2) Show that

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x) \quad -(1)$$

we know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Differentiating (1) partially w.r.t x we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum P_n'(x) t^n \quad - (\cancel{11})$$

i.e $$t(1-2tx+t^2)^{-3/2} = \sum P_n'(x) t^n \quad - (11)$$

Again differentiating (1) partially w.r.t t we have

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum n P_n(x) t^{n-1} \quad - (111)$$

by (111)/(11) $$\frac{x-t}{t} = \frac{\sum n P_n(x) t^{n-1}}{\sum P_n'(x) t^n}$$

i.e. $\boxed{nP_n = xP_n'(x) - P_{n-1}'(x)}$ proved //

Formula III $(2n+1)P_n = P_{(n+1)}'(x) - P_{(n-1)}'(x)$

we know that

$$(n+1)P_{n+1}(x) = (2n+1)xP_n - nP_{n-1} \quad -(1)$$

Diff (1) w.r.t x

$$(n+1)P_{(n+1)}' = (2n+1)P_n + (2n+1)xP_n' - nP_{n-1}' \quad -(2)$$

by formula 2

$$xP_n' - P_{n-1}' = nP_n \quad - (3)$$

use (3) in (2)

$$(n+1)P_{(n+1)}' = (2n+1)P_n + (2n+1)\left[nP_n + P_{n-1}'\right] - nP_{n-1}'$$

$$\Rightarrow \boxed{(2n+1)P_n = P_{(n+1)}' - P_{(n-1)}'}$$

Formula IV $\boxed{P_n' - xP_{n-1}' = nP_{n-1}}$

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1} \quad \text{formula} - 1$$

Replacing n by $n-1$

$$nP_n = (2n-1)xP_{n-1} - \cancel{(n+1)}(n-1)P_{n-2}$$

Diff w.r.t x

$$nP_n' = (2n-1)P_{n-1} + (2n-1)xP_{(n-1)}' - (n-1)P_{n-2}'$$

$$\Rightarrow n\left[P_n' - xP_{n-1}'\right] - (n-1)\left[xP_{n-1}' - P_{n-2}'\right] = (2n-1)P_{n-1}$$

$$\Rightarrow n\left[P_n' - xP_{n-1}'\right] - (n-1)\left[(n-1)P_{n-1}\right] = (2n-1)P_{n-1}$$

$$\Rightarrow n\left[P_n' - xP_{n-1}'\right] = \left[(n-1)^2 + (2n-1)\right]P_{n-1} = n^2P_{n-1}$$

$$\Rightarrow \boxed{P_n' - xP_{n-1}' = nP_{n-1}}$$

Formula V $(x^2-1)P_n' = n\left[xP_n - P_{n-1}\right]$

Soln:- by III $\quad P_n' - xP_{n-1}' = nP_{n-1} \quad -(1)$

$$xP_n' - P_{n-1}' = nP_n \quad \text{by II} \quad -(2)$$

multiplying (2) by x and substracting from (1), we get

11 ~~we get~~ ~~$1-x^2$~~

we get $(1-x^2)P_n' = n(P_{n-1} - xP_n)$

$\boxed{\text{Formula VI}}$ $(x^2-1)P_n' = (n+1)(P_{n+1} - xP_n)$

Soln:- we know that

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1} \text{ [by for I]}$$

which can be written as

$$(n+1)(P_{n+1} - xP_n) = n(xP_n - P_{n-1}) \quad -(1)$$

but $(x^2-1)P_n' = n(xP_n - P_{n-1})$ $-(2)$ $\boxed{\text{by formula V}}$

from (1) and (2)

$$\boxed{(x^2-1)P_n' = (n+1)(P_{n+1} - xP_n)}$$

proved