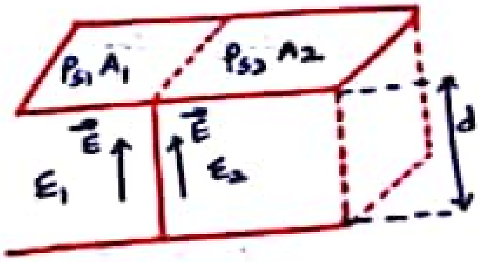
DIELECTRIC BOUNDARY NORMAL TO THE PLATES:

Consider the composite capacitor in which dielectric boundary is normal to the conducting plates.



The dielectric ϵ_1 occupying area A_1 of the plates, while dielectric ϵ_2 occupying area A_2 as shown in the figure.

The total potential across the two plates is V and the distance b/w the plates is d. Hence magnitude of $\vec{E}$ is

$$E = \frac{V}{d}$$

$\therefore$ At the boundary, both $\vec{E_1}$ and $\vec{E_2}$ are tangential and for dielectric - dielectric interface tangential components are equal.

$$E_{tan1} = E_{tan2} = E_1 = E_2 = \frac{V}{d} \quad \text{--- ①}$$

Now $D_1 = \epsilon_1 E_1$ & $D_2 = \epsilon_2 E_2$ ---→ ②

sub ① in ②,

$$D_1 = \frac{\epsilon_1 V}{d} \quad \text{-----} \quad D_2 = \frac{\epsilon_2 V}{d} \quad \text{--- ③}$$

on the plates the charge is divided into two parts

on area A_1, the charge density is $\rho_{s1} = D_1$ while
on area A_2, the charge density is $\rho_{s2} = D_2$ } — ④

$$\therefore Q = Q_1 + Q_2 \quad \text{--- ⑤}$$

$$= \rho_{s1} A_1 + \rho_{s2} A_2 \quad \text{--- ⑥}$$

sub ④ in ⑥

$$= D_1 A_1 + D_2 A_2 \quad \text{--- ⑦}$$

sub ③ in ⑦

$$= \frac{\epsilon_1 V A_1}{d} + \frac{\epsilon_2 V A_2}{d}$$

$$C = \frac{Q}{V} = \frac{\frac{\epsilon_1 V A_1 + \epsilon_2 V A_2}{d}}{V} \Rightarrow \frac{\epsilon_1 A}{d} + \frac{\epsilon_2 A}{d}$$

$$\boxed{C = C_1 + C_2}$$

where $C_1 = \frac{\epsilon_1 A}{d}$ $C_2 = \frac{\epsilon_2 A}{d}$.

Thus is dielectric boundary is $\parallel^{el}$ to the plates the arrangement is equivalent to two capacitors in series for which

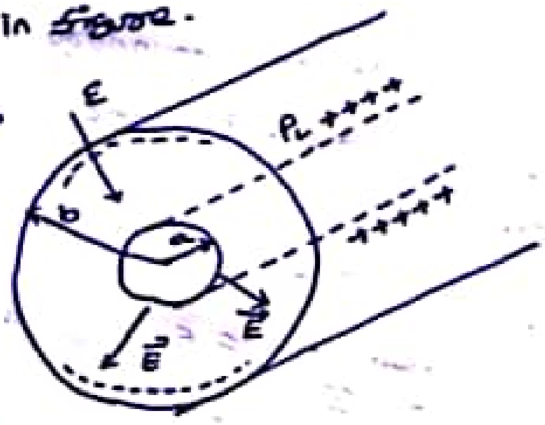
$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

while is the dielectric boundary is normal to the plates, the arrangement is equivalent to two capacitors in $\parallel^{el}$ for which

$$C_{eq} = C_1 + C_2 .$$

CAPACITANCE OF A CO-AXIAL CABLE:

consider a co-axial cable or co-axial capacitor as shown in figure.



The two concentric conductors are separated by dielectric of permittivity ε.

The length of the cable is L meters.

The inner conductor carries a charge density $+\rho_L$ c/m on its surface then equal and opposite charge density $-\rho_L$ c/m exists on the outer conductor.

$$\therefore \quad Q = \rho_L \times L \quad ---- ①$$

Assuming cylindrical co-ordinate system, $\vec{E}$ will be radial from inner to outer and for infinite line charge it is given by

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon r} \vec{a_r} \quad --- ②$$

$\vec{E}$ is directed from inner conductor to outer conductor. The potential difference is work done in moving unit charge against $\vec{E}$ i.e. from $r=b$ to $r=a$.

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To find potential difference, consider $\vec{dL}$ in radial direction which is $dr\,\vec{a_r}$.

$$\therefore \vec{dL} = dr\,\vec{a_r} \quad \text{----} (3)$$

$$\therefore V = -\int_{-}^{+} \vec{E}\cdot\vec{dL}$$

$$= -\int_{-}^{+} \frac{\rho_L}{2\pi\varepsilon r}\vec{a_r}\cdot dr\,\vec{a_r} = -\frac{\rho_L}{2\pi\varepsilon}\left[\ln r\right]_b^a$$

$$= -\frac{\rho_L}{2\pi\varepsilon}\ln\left[\frac{a}{b}\right]$$

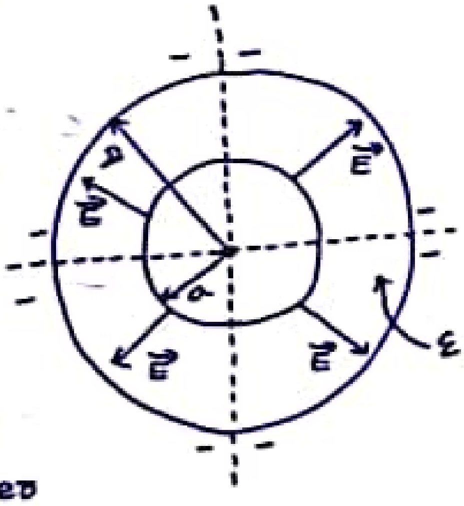
$$\boxed{\therefore V = \frac{\rho_L}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right)}$$

$$\therefore C = \frac{Q}{V} = \frac{\rho_L L}{\frac{\rho_L}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}\ F$$

$$\boxed{C = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}\ F} \quad \text{----} (4)$$

SPHERICAL CAPACITOR:-

Consider a spherical capacitor formed of two concentric spherical conducting shells of radius 'a' and 'b'. The capacitor is shown in figure.



The radius of outer sphere is 'b' while that of inner sphere is 'a'. Thus $b > a$. The region b/w the two spheres is filled with a dielectric of permittivity ε.

The inner sphere is given a +ve charge (+Q) while for the outer sphere it is (−Q).

considering, Gaussian surface as a sphere of radius r, it can be obtained that $\vec{E}$ is in radial direction and given by.

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2} \vec{a_r} \ V/m. \quad \text{----} ①$$

[Elementary Equation]

The potential difference is work done in moving unit positive charge against the direction of $\vec{E}$ i.e from $r=b$ to $r=a$

$$\therefore \ V = -\int_{-}^{+} \vec{E}\cdot\vec{dL} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2} \vec{a_r}\cdot\vec{dL} \quad \text{---} ②$$

$$\vec{dL} = dr\,\vec{a_r} \quad \text{---} ③$$

sub ③ in ② we get

$$V = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2} \vec{a_r}\cdot dr\,\vec{a_r}$$

$$\cdot -\int_{b}^{a} \frac{Q}{4\pi\varepsilon r^2} dr = -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_b^a$$

$$\boxed{V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right] \text{ Volt}} \quad \text{---} ④$$

$$\therefore \ \text{Now} \quad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right]}$$

$$\boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]} \ F} \quad \text{---} ⑤$$

CAPACITANCE OF SINGLE ISOLATED SPHERE:

Consider a single isolated sphere of radius 'a' given a charge of +Q. It forms a capacitance with an outer plate which is infinitely large hence $b = \infty$.

The capacitance of such a single isolated spherical conductor can be obtained by substituting $b=\infty$ in above eqn. (5)

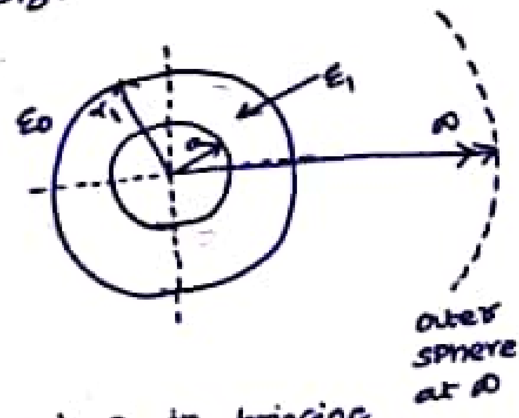
$$\therefore \quad C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{\infty}} \qquad \text{but } \frac{1}{\infty} = 0$$

$$\boxed{C \Rightarrow 4\pi\epsilon a \text{ farads}}$$

(stray capacitance of an isolated body)

ISOLATED SPHERE COATED WITH DIELECTRIC:

Consider a single isolated sphere coated with a dielectric having permittivity ϵ_1, upto radius r_1. The radius of inner sphere is 'a' as shown in fig.



It is placed in a free space so outside sphere $\epsilon = \epsilon_0$. It carries a charge of +Q.

So for $a < r < r_1$, $\epsilon = \epsilon_1$.

for $r > r_1$, $\epsilon = \epsilon_0$.

The potential difference is workdone in bringing unit positive charge from outer sphere $r=\infty$ to inner sphere $r=a$ against $\vec{E}$. This is to be splitted into two as

$$V = -\int_{-}^{+} \vec{E}\cdot\vec{dL} = -\int_{r=\infty}^{r=a} \vec{E}\cdot\vec{dL}$$

$$= -\int_{r=\infty}^{r=r_1} \vec{E}\cdot\vec{dL} - \int_{r=r_1}^{r=a} \vec{E}\cdot\vec{dL} \quad \text{---} (1)$$

Now for $a < r < r_1$ for $r_1 < r < \infty$

$$\vec{E}_1 = \frac{Q}{4\pi\varepsilon_1 r^2}\vec{a}_r \qquad \vec{E}_2 = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$

while $\vec{dL} = dr\,\vec{a}_r$

$\therefore$ Eqn ① becomes,

$$V = -\int_{\infty}^{r_1}\frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r \cdot dr\,\vec{a}_r - \int_{r_1}^{a}\frac{Q}{4\pi\varepsilon_1 r^2}\vec{a}_r \cdot dr\,\vec{a}_r$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\int_{\infty}^{r_1}\frac{1}{r^2}dr + \frac{1}{\varepsilon_1}\int_{r_1}^{a}\frac{1}{r^2}dr\right]$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{r_1} + \frac{1}{\varepsilon_1}\left[-\frac{1}{r}\right]_{r_1}^{a}\right]$$

$$V = -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left(-\frac{1}{r_1}+\frac{1}{\infty}\right) + \frac{1}{\varepsilon_1}\left[-\frac{1}{a}+\frac{1}{r_1}\right]\right]$$

$$V = \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left(\frac{1}{r_1}\right) + \frac{1}{\varepsilon_1}\left(\frac{1}{a}\right) - \frac{1}{\varepsilon_1}\left(\frac{1}{r_1}\right)\right]$$

$$V = \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right] \text{ Volts}$$

$$\therefore \quad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right)+\frac{1}{\varepsilon_0 r_1}\right]}$$

$$C = \frac{4\pi}{\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right)+\frac{1}{\varepsilon_0 r_1}\right]} \Rightarrow \frac{1}{C} = \frac{\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right)+\frac{1}{\varepsilon_0 r_1}}{4\pi}$$

$$\Rightarrow \frac{1}{C} = \frac{\frac{1}{a}-\frac{1}{r_1}}{4\pi\varepsilon_1} + \frac{1}{4\pi\varepsilon_0 r_1}$$

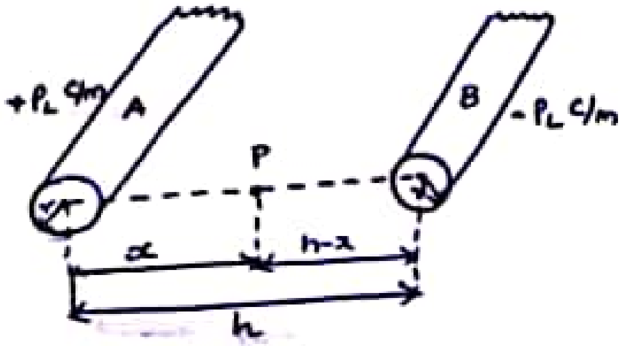
Let $C_1 = \frac{4\pi\varepsilon_1}{\frac{1}{a}-\frac{1}{r_1}}$ & $C_2 = 4\pi\varepsilon_0 r_1$

$$\therefore \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \qquad \boxed{C = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}}$$

CAPACITANCE BETWEEN TWO TRANSMISSION LINES:

Let us consider two $\parallel^{el}$ conductors A and B of radius 'r' seperated by a distance 'h'. If A has charge $+\rho_L$ c/m along its length, it will induce $-\rho_L$ c/m on conductor, B. At any Point P at a distance x from the centre of A, electric field intensity due to A is.

$$\vec{E}_1 = \frac{\rho_L}{2\pi\varepsilon x}\,\vec{a}_x$$



Electric field intensity at P due to B is

$$\vec{E}_2 = \frac{-\rho_L}{2\pi\varepsilon(h-x)}\,(+\vec{a}_x)$$

The Total field Intensity at P is.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_L}{2\pi\varepsilon}\left[\frac{1}{x} + \frac{1}{h-x}\right]\vec{a}_x$$

Potential rise from B to A

$$V = -\int_B^A \vec{E}\cdot\vec{dl}$$

At the surface of A, $x = r$
B, $x = h-r$.

$$\therefore V = -\int_{x=h-r}^{x=r} \frac{\rho_L}{2\pi\varepsilon}\left[\frac{1}{x} + \frac{1}{h-x}\right]dx = -\frac{\rho_L}{2\pi\varepsilon}\left[\ln x - \ln(h-x)\right]_{h-r}^{r}$$

$$V = \frac{-\rho_L}{2\pi\varepsilon}\left[\ln(r) - \ln(h-r) - \ln(h-r) + \ln\left(h-(h-r)\right)\right]$$

$$V = -\frac{\rho_L}{2\pi\varepsilon}\left[2\ln(r) - 2\ln(h-r)\right] = \frac{\rho_L}{\cancel{2}\pi\varepsilon}\cancel{2}\left[\ln(r) - \ln(h-r)\right]$$

$$\therefore V = \frac{\rho_L}{\pi\varepsilon}\left[\ln\left(\frac{h-r}{r}\right)\right]$$

$$\therefore C = \frac{Q}{V} = \frac{\rho_L \cdot l}{\frac{\rho_L}{\pi\varepsilon}\ln\left(\frac{h-r}{r}\right)} = \frac{\pi\varepsilon l}{\ln\left(\frac{h-r}{r}\right)}$$

$$\boxed{\therefore C = \frac{\pi\varepsilon l}{\ln\left(\frac{h-r}{r}\right)}}$$

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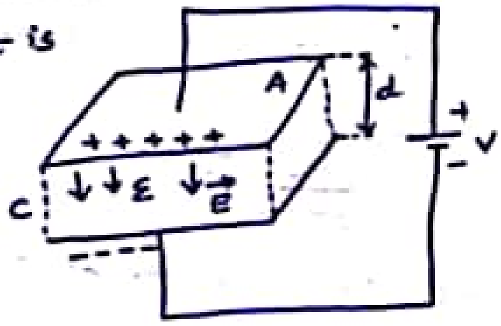
ENERGY STORED IN A CAPACITOR:

It is seen that capacitors can store the energy. Let's find the expression for the energy stored in a capacitor.

Consider a $\parallel^{el}$ plate capacitor as shown in the figure. It is supplied with voltage V.



Let $\vec{a}_N$ is the direction normal to the plates

$$\therefore \vec{E} = \frac{V}{d}\vec{a}_N \quad \text{---} ①$$

The energy stored is given by,

$$W_E = \frac{1}{2}\int_{Vol} \vec{D}.\vec{E}\, dv$$

$$= \frac{1}{2}\int_{Vol} \varepsilon \vec{E}.\vec{E}\, dv \qquad \text{but } \vec{E}.\vec{E} = |\vec{E}|^2$$

$$= \frac{1}{2}\int_{Vol} \varepsilon |\vec{E}|^2 dv \qquad \text{but } |\vec{E}| = \frac{V}{d}$$

$$= \frac{1}{2}\varepsilon \frac{V^2}{d^2}\int_{Vol} dv \qquad \text{but } \int_{Vol} dv = \text{volume} = A \times d$$

$$= \frac{1}{2}\varepsilon \frac{V^2 A d}{d^2}$$

$$W_E = \frac{1}{2}\frac{\varepsilon A}{d}V^2 = \frac{1}{2}CV^2 \quad \left[\because C = \frac{\varepsilon A}{d}\right]$$

ENERGY DENSITY:

Energy density is energy stored per unit volume as volume tends to zero.

$$\therefore W_E = \frac{1}{2}\varepsilon \int_{Vol} |\vec{E}|^2 dv$$

$$W_E = \frac{1}{2}\varepsilon |\vec{E}|^2 \ J/m^3 = \text{Energy density.}$$

Using $|\vec{D}| = \varepsilon|\vec{E}|$ in above expression.

$$W_E = \frac{1}{2}\frac{|\vec{D}|^2}{\varepsilon} = \frac{1}{2}|\vec{D}|\,|\vec{E}| \ J/m^3.$$

POISSON'S & LAPLACE'S EQUATIONS:

From the Gauss's law in the point form, Poisson's equation can be derived.

consider the Gauss's law in the point form as

$$\nabla \cdot \vec{D} = \rho_v \quad \text{---} (1)$$

flux density — $\vec{D}$; ρ_v — volume charge density

W.K.T

$$\vec{D} = \varepsilon \vec{E} \quad \text{---} (2)$$

sub (2) in (1) we get

$$\nabla \cdot \varepsilon \vec{E} = \rho_v \quad \text{---} (3)$$

From the gradient relationship

$$\vec{E} = -\nabla V \quad \text{---} (4)$$

substitute (4) in (3) we get

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v$$

$$-\varepsilon[\nabla \cdot \nabla V] = \rho_v$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon} \quad \text{---} (5)$$

But $\nabla \cdot \nabla = \nabla^2$

$$\therefore (5) \Rightarrow \boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \quad \text{---} (6)$$

Equation (6) is called Poisson's Equation.

If in certain region, $\rho_v = 0$, which is true for dielectric medium then Poisson's Equation takes a form

$\boxed{\nabla^2 V = 0}$ (for charge free region).

This is a special case of Poisson's equation called as Laplace's Equation.

∇^2 is called Laplacian of V.

UNIQUENESS THEOREM:-

The boundary value Problems can be solved by number of methods such as analytical, graphical, experimental etc.

Thus there is a question that, is the solution of Laplace's equation solved by any method, unique? The answer to this question is the Uniqueness theorem, which is Proved by contradiction method.

Assume that the Laplace's equation has two solutions say V_1 and V_2, both are functions of the co-ordinates of the system used. These solutions must satisfy Laplace's equation. So we can write,

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 \text{ ---- (1)}$$

Both the solutions must satisfy the boundary conditions as well. At the boundary, the Potentials at different points are same due to equipotential surface then,

$$V_1 = V_2 \text{ --- (2)}$$

Let the difference b/w the two solutions is V_d

$$\therefore \quad V_d = V_2 - V_1 \text{ --- (3)}$$

Using Laplace's equation for the difference V_d,

$$\nabla^2 V_d = \nabla^2 (V_2 - V_1) = 0 \text{ ---- (4)} \Rightarrow \nabla^2 V_2 - \nabla^2 V_1 = 0 \text{ --- (5)}$$

on the boundary $V_d = 0$ [from (2) & (3)]

from divergence Theorem,

$$\int_{Vol} (\nabla \cdot \vec{A})\, dv = \oint_S \vec{A} \cdot \vec{ds} \text{ --- (6)}$$

Let $\vec{A} = V_d \nabla V_d$ and from vector algebra

$$\nabla \cdot (\alpha \vec{B}) = \alpha (\nabla \cdot \vec{B}) + \vec{B} \cdot (\nabla \alpha) \text{ --- (7)}.$$

Now use this for $\nabla\cdot(V_d \nabla V_d)$ with $\alpha = V_d$ and $\nabla V_d = \vec{B}$

$$\nabla\cdot(V_d \nabla V_d) = V_d(\nabla\cdot\nabla V_d) + \nabla V_d\cdot(\nabla V_d)$$

But $\nabla\cdot\nabla = \nabla^2$ hence

$$\nabla\cdot(V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d\cdot\nabla V_d \text{ ---} (8)$$

using eqn (4) in (8) ie $\nabla^2 V_d = 0$ we get

$$\nabla\cdot(V_d \nabla V_d) = \nabla V_d\cdot\nabla V_d \text{ ----} (9)$$

To use this in equation (6)

Let $V_d \nabla V_d = \vec{A}$ hence

$$\nabla\cdot(V_d \nabla V_d) = \nabla\cdot\vec{A} = \nabla V_d\cdot\nabla V_d$$

$$\int_{Vol} \nabla V_d\cdot\nabla V_d \, dv = \oint_S V_d \nabla V_d\cdot\vec{ds} \text{ ---} (10)$$

But $V_d = 0$ on boundary, hence RHS of (10) becomes zero

$$\therefore \int_{Vol} (\nabla V_d\cdot\nabla V_d) dv = 0$$

$$\int_{Vol} |\nabla V_d|^2 dv = 0 \text{ as } \nabla V_d \text{ is a vector ---} (11)$$

Now Integration can be zero under two conditions,

(i) the quantity under Integral sign is zero

(ii) the quantity is +ve in some regions and -ve in some other regions by equal amount and hence zero.

$$\therefore |\nabla V_d|^2 = 0$$

$$\nabla V_d = 0 .$$

As the gradient of $V_d = V_2 - V_1$ is zero means $V_2 - V_1$ is constant and not changing with any co-ordinates.

But considering boundary it can be proved that

$V_2 - V_1 = const = zero.$

$V_2 = V_1$

This proves that both the solutions are equal and cannot be different.

UNIQUENESS THEOREM states that

If the solution of Laplace's equation satisfies the boundary condition then that solution is unique, by whatever method it is obtained.

TUTORIAL PROBLEMS

1. Verify that the potential field given below satisfies the Laplace's equation $V = 2x^2 - 3y^2 + z^2$

Given field is in cartesian system

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}[2x^2 - 3y^2 + z^2] + \frac{\partial^2}{\partial y^2}[2x^2 - 3y^2 + z^2] + \frac{\partial^2}{\partial z^2}[2x^2 - 3y^2 + z^2]$$

$$= \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial y}[-6y] + \frac{\partial}{\partial z}[2z]$$

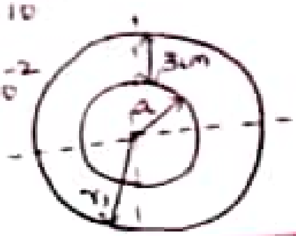
$$\Rightarrow 4 - 6 + 2 = 0$$

$\boxed{\nabla^2 V = 0}$ Thus the field satisfies the Laplace's Equation

2. Find the capacitance of a conducting sphere of 2cm in diameter, covered with a layer of polyethelene with $\varepsilon_r = 2.26$ and 3cm thick

a = radius of sphere $= \frac{d}{2} = 1\,cm = 1 \times 10^{-2}$

$r_1 = a +$ thickness $= 1 + 3 = 4\,cm = 4 \times 10^{-2}$



$$\therefore C = \frac{4\pi}{\frac{1}{\varepsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}} \; F$$

$$= \frac{4\pi}{\frac{1}{2.26}\left[\frac{1}{1 \times 10^{-2}} - \frac{1}{4 \times 10^{-2}}\right] + \frac{1}{8.854 \times 10^{-12} \times 4 \times 10^{-2}}}$$

$\boxed{C = 1.9121\ PF}$

3. A coil of 500 turns is wound on a closed iron ring of mean radius 10 cm and cross section area of 3 cm². Find the self inductance of the winding if the relative permeability of iron is 800.

$\mu_r = 800$

$A = 3\,cm^2 = 3 \times 10^{-4}\,m^2$

$R = 10\,cm = 10 \times 10^{-2}\,m$

$N = 500$

$L = \frac{\mu N^2 A}{2\pi R}$

$\mu = \mu_0 \mu_r$

$\mu_0 =$

$L =$

$\boxed{L =}$