1 Concept of Numerical Integration (EE+ECE) (Module-3)

Newton's cotes formula

By Newton's forward interpolation formula

$$f(x) = f(x_0+rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{\lfloor 2}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{\lfloor 3}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{\lfloor 4}\Delta^4 y_0 + \cdots$$

Here $x = x_0 + rh$, $\frac{x-x_0}{h} = r$

if $x = x_0$: $x_0 = x_0 + rh \Rightarrow r = 0$

if $x = x_n$: $x_n = x_0 + nh$, $\frac{x_n - x_0}{h} =$

$$I = \int_a^b f(x)\,dx$$ where $f(x)$ takes the values $y_0, y_1, y_2, \cdots y_n$ for $x_0, x_1, x_2, \cdots x_n$

Let us divide the interval $[a, b]$ into $n$ equal parts of width $h$

$x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$, $x_n = x_0 + nh = b$

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = h\int_0^n f(x_0+rh)\,dr$$

let $x = x_0 + rh$, $dx = h\,dr$

if $x = x_0$, thus $rh = 0 \Rightarrow r = 0$

if $x_n = x$, $x_n = x_0 + rh$

$x_0 + nh = x_0 + rh$

$\Rightarrow nh = rh$

$\Rightarrow n = r$

→ Newton's formula

$$\Rightarrow h\int_0^n \left[f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{\lfloor 2}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{\lfloor 3}\Delta^3 f(x_0) + \cdots\right] dr$$

Integrating term by term

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

(2) eq^n (1) is known as Newton's cote's quadrature formula

in eq (1) if we put $n=1$ we get Trapezoidal rule

(ii) if we put $n=2$ we get Simpson's $1/3$ rule

(iii) if we put $n=3$ we get " $3/8$ rule

respectively

Now For Trapezoidal rule put $n=1$ in (1)

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = h\left[y_0 + \frac{1}{2}\left(y_1 - y_0\right)\right]$$

$$= \frac{h}{2}\left[y_1 - y_0\right]$$

IIIy

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}\left[y_2 - y_1\right] \quad \text{and} \quad \int_{x_0+2h}^{x_0+3h} = \frac{h}{2}\left[y_3 - y_2\right]$$

IIIy

$$\int_{x_0+(n-1)h}^{x_0+nh} = \frac{h}{2}\left[y_n - y_{n-1}\right]$$

Thus

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots = \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right)\right]}$$

In Simpson's $\frac{1}{3}$ rule Put $n=2$ in (1)

thus $$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

IIly $$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}\left[y_{n-2} + y_{n-1} + y_n\right]$$

adding above we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4\underbrace{(y_1+y_3+\cdots y_{n-1})}_{\text{odd terms}} + 2\underbrace{(y_2+y_4+\cdots + y_{n-2})}_{\text{even terms}}\right]$$

[Note:– In Simpson's rule we have to divide given interval into even numbers of equal sub intervals i.e. ~~or~~ no of ordinates is odd]

$\boxed{\text{Simpson's } \frac{3}{8} \text{ rule}}$

Put $n=3$ in quotes formula

divide the interval $[a,b]$ ~~into odd number~~ of sub-intervals (i.e no of ~~intervals~~ ordinates = even)

multiple of 3

Thus $$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

④

Similarly $\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}\left(y_3+3y_4+3y_5+y_6\right)$

in last $\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right)$

adding all above we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left\{(y_0+y_n)+3\underbrace{(y_1+y_2+y_4 - +y_{n-1})}_{\text{other than multiple of 3}} + 2\underbrace{(y_3+y_6+ - +y_{n-3})}_{\text{multiple of 3}}\right\}$$

Note! – Only ~~these~~ above three methods is for you according to your syllabus

Ex→1 Evaluate $I = \int_0^6 \frac{1}{1+x}\,dx$ using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$ rule, (iii) Simpson's $\frac{3}{8}$ rule

Soln: – ① In Trapezoidal rule there is no condition of taking Sub-intervals

(ii) In Simpson's $\frac{1}{3}$ rule taking even number of Sub-intervals

(iii) In Simpson's $\frac{3}{8}$ rule taking multiple of 3 = no of sub-intervals

thus no of Sub-intervals = 6, 12 — are both including Simpson's $\frac{1}{3}$ rule and $\frac{3}{8}$ rule both]

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(8)

So that divide the given interval [0, 6] into six equal sub-intervals each of width is

1 means 7 numbers of ordinates

→ 6 subintervals

| NOCO | $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| | $y = \frac{1}{1+x}$ | 1 $y_0$ | 0.5 $y_1$ | $\frac{1}{3}$ $y_2$ | $\frac{1}{4}$ $y_3$ | $\frac{1}{5}$ $y_4$ | $\frac{1}{6}$ $y_5$ | $\frac{1}{7}$ $y_6$ |

→ 7 number ordinates

(I)

∴ By Trapezoidal rule $\quad h = \frac{6-0}{6} = 1$

$$\int_0^6 \frac{dx}{1+x} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{7}\right)+2\left(0.5+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\right] = 2.02142857$$

(II) By Simpson's one-third rule

$$\frac{h}{3}\left[(y_0+y_6)+4(\underbrace{y_1+y_3+y_5}_{\text{odd entry}})+2(\underbrace{y_2+y_4}_{\text{even entry}})\right]$$

$$= \frac{h}{3}\left[\left(1+\frac{1}{7}\right)+4\left(0.5+\frac{1}{4}+\frac{1}{6}\right)+2\left(\frac{1}{3}+\frac{1}{5}\right)\right] = 1.95873016$$

(III) By Simpson's $\frac{3}{8}$ rule

$$= \frac{3h}{8}\left[(y_0+y_6)+3(\underbrace{y_1+y_2+y_4+y_5}_{\text{other than multiple of 3}}) + 2(y_3)\right]$$

$$= \frac{3h}{8}\left[\left(1+\frac{1}{7}\right)+3\left(\frac{1}{2}+\frac{1}{3}+\cancel{\frac{1}{4}}+\frac{1}{5}\right)+2\times\frac{1}{4}+\frac{1}{6}\right] = 1.96607143$$

Actual integration $\int_0^6 \frac{1}{1+x}dx = [\log(1+x)]_0^6 = \log_e 7$

$= 1.94591015$

Error in every three methods are

(I) in Trapezoidal rule $2.02142857 - 1.94591015$
$= 0.07551842$

(II) in Simpson's $\frac{1}{3}$ rule $= 2.02142857 - 1.95873016$
$= 0.06269841$

(III) in Simpson's $\frac{3}{8}$ rule
$= 2.02142857 - 1.96607143$
$= 0.05535714$

By above ~~condition~~ comparison between three methods we see that
Simpson's $\frac{3}{8}$ rule gives least error from actual value
and Trapezoidal give much more error from actual value.