If the vector field

$$\bar{F} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$$ is irrotational

find a, b, c and determine ϕ such that $\bar{F} = \nabla\phi$.

M3 Civil June 2017

sol^n :-

21:20

Sol^n :-

STEP 1 :-

$\bar{F}$ is irrotational i.e Curl $\bar{F}=0$

STEP 2 :-

Curl $\bar{F}=0$

$$\phi = \int_{\substack{y,z \\ \text{Const}}} f_1\,dx + \int_{\substack{z \\ \text{const}}} \underset{x}{f_2 \text{ free from}}\,dy + \int \underset{x \text{ and } y}{f_3 \text{ free from}}\,dz$$

$$\begin{vmatrix} f_1 & f_2 & f_3 \end{vmatrix}$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\bar{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$

$$-\bar{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

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$$+\bar{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$\bar{i}[c-(-1)] - \bar{j}[4-a] + \bar{k}[b-2] = 0$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$

Compairing componentwise we get

$c-(-1) = 0 \Rightarrow c+1 = 0 \Rightarrow c = -1$

$4 - a = 0 \Rightarrow a = 4$

$b - 2 = 0 \Rightarrow b = 2$

Substituting in $\bar{F}$ we get

$$\bar{F} = (x+2y+4z)\,\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\,\bar{k}$$

To find ϕ

$$\phi = \int (x+2y+4z)\,dx + \int \underline{(2x-3y-z)}\,dy + \int (4x-y+2z)\,dz + C$$

Put $x=0$

Put $\underline{x=0}$, $y=0$

$$\phi = \int (x+2y+4z)\,dx + \int (-3y-z)\,dy + \int 2z\,dz + c$$

$$= \frac{x^2}{2} + 2yx + 4zx - \frac{3y^2}{2} - zy + \frac{2z^2}{2} + c$$

$$= \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - zy + z^2 + c$$

$$= \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - zy + z^2 + C$$

STEP 4 :-

Required values are

$a = 4$, $b = 2$ and $c = -1$ and

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - zy + z^2 + C$$