VECTOR CALCULUS - Divergence of Curl

Ex : Prove that div (curl $\vec{f}$) is zero

$$\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k} \qquad \text{div}\,\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{curl}\,\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(f_3) - \frac{\partial}{\partial z}(f_2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(f_3) - \frac{\partial}{\partial z}(f_1)\right] + \vec{k}\left[\frac{\partial}{\partial x}(f_2) - \frac{\partial}{\partial y}(f_1)\right]$$

$$\text{div}(\text{curl}\,\vec{f}) = \frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] + \frac{\partial}{\partial y}\left[-\frac{\partial f_3}{\partial x} + \frac{\partial f_1}{\partial z}\right] + \frac{\partial}{\partial z}\Big[$$

Curl of Grad $\phi = 0$

$$\vec{\nabla} \times (\vec{\nabla}\phi) = 0$$

$$\vec{\nabla} \times (\vec{\nabla}\phi) = \vec{\nabla} \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \right)$$

$$\nabla \times (\nabla \phi) = 0$$

$$\vec{\nabla} \times (\vec{\nabla}\phi) = \vec{\nabla} \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$\left| \quad \frac{\partial \phi}{\partial x} \qquad \frac{\partial \phi}{\partial y} \qquad \frac{\partial \phi}{\partial z} \quad \right|$$

$$= \hat{i}\left\{ \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) \right\} + \hat{j}\left\{ \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) \right\}$$

$$+ \hat{k}\left\{ \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) \right\}$$

$$= \hat{i}\left\{ \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right\} + \hat{j}\left\{ \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z} \right\}$$

$$+ \hat{k}\left\{ \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right\}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right\}$$

$$= \hat{i}\left\{\cancel{\frac{\partial^2 \phi}{\partial y \partial z}} - \cancel{\frac{\partial^2 \phi}{\partial z \partial y}}\right\} + \hat{j}\left\{\cancel{\frac{\partial^2 \phi}{\partial z \partial x}} - \cancel{\frac{\partial^2 \phi}{\partial x \partial z}}\right\}$$

$$+ \hat{k}\left\{\cancel{\frac{\partial^2 \phi}{\partial x \partial y}} - \cancel{\frac{\partial^2 \phi}{\partial y \partial x}}\right\}$$

$$= \hat{i}\cdot 0 + \hat{j}\cdot 0 + \hat{k}\cdot 0$$

$$= 0$$

$$\left| \; \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \right.$$

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