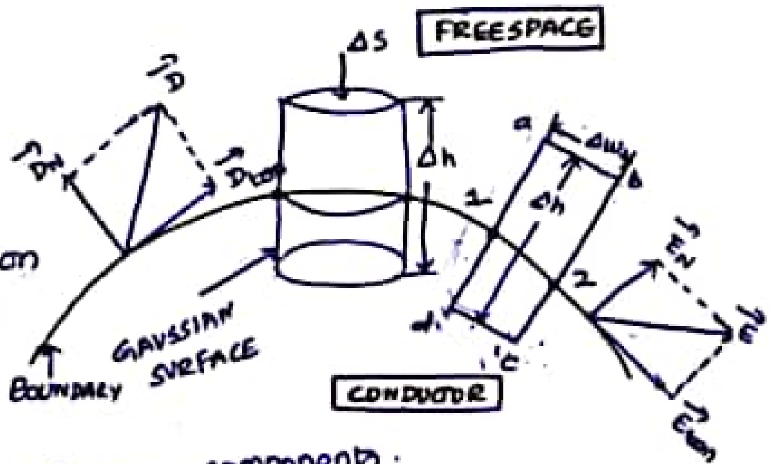
To determine the boundary conditions let us use the closed path and the Gaussian surface.

Consider the conductor free space boundary as shown in fig below.



$\vec{E}$ at BOUNDARY:

Let $\vec{E}$ be the Electric field intensity, in the direction shown in the figure, making some angle with the boundary.

This $\vec{E}$ can be resolved into two components.

1. The component tangential to the surface $(\vec{E}_{tan})$
2. The component normal to the surface $[\vec{E}_N]$

It is known that

$$\oint \vec{E} \cdot \vec{dL} = 0$$

The integral $\oint \vec{E} \cdot \vec{dL}$ carried over a closed contour is zero. i.e work done in carrying a unit +ve charge along a closed path is zero.

Consider a rectangular closed path abcda as shown in fig. It is traced in clockwise direction as a-b-c-d-a and hence $\oint \vec{E} \cdot \vec{dL}$ can be divided into four parts.

$$\oint \vec{E} \cdot \vec{dL} = \int_a^b \vec{E} \cdot \vec{dL} + \int_b^c \vec{E} \cdot \vec{dL} + \int_c^d \vec{E} \cdot \vec{dL} + \int_d^a \vec{E} \cdot \vec{dL} = 0.$$

The closed contour is placed in such a way that its two sides a-b and c-d are $\parallel^{el}$ to tangential direction to the surface while the other two are normal to the surface, at the boundary.

The rectangle is an elementary rectangle with elementary height Δh and elementary width Δw. The rectangle is place in such a way that half of it is in the conductor and remaining half is in the free space.

Thus $\Delta h/2$ is in the conductor and $\Delta h/2$ is in the free space.

Now the portion c-d is in the conductor where $\vec{E}=0$ hence the corresponding integral is zero

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} + \int_b^c \vec{E}\cdot\vec{dL} + \int_d^a \vec{E}\cdot\vec{dL} = 0 \quad ---- \text{(A)}$$

As the width Δw is very small, $\vec{E}$ over it can be assumed constant and hence can be taken out for integration.

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} = \vec{E}\int_a^b \vec{dL} = \vec{E}(\Delta w)$$

But Δw is along tangential direction to the boundary in which direction $\vec{E} = \vec{E}_{tan}$

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} = E_{tan}(\Delta w) \quad \text{where } E_{tan} = |E_{tan}| \quad --- (1)$$

Now b-c is $\parallel^{d}$ to the normal components so we have $\vec{E} = \vec{E}_N$ along this direction, Let $E_N = |\vec{E}_N|$

over the small height Δh, E_N can be assumed constant and can be taken out of integration.

$$\therefore \int_b^c \vec{E}\cdot\vec{dL} = \vec{E}\int_b^c \vec{dL} = \vec{E}_N \int_b^c \vec{dL}$$

But out of b-c, b-2 is in free space and 2-c is in the conductor where $\vec{E}=0$

$$\therefore \int_b^c \vec{dL} = \int_b^2 \vec{dL} + \int_2^c \vec{dL} = \frac{\Delta h}{2} + 0 = \frac{\Delta h}{2}$$

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Let the area of top and bottom is same equal to Δs

$$\therefore \int_{top} \vec{D}\cdot\vec{ds} + \int_{bottom} \vec{D}\cdot\vec{ds} + \int_{lateral} \vec{D}\cdot\vec{ds} = Q \quad \text{---- (1)}$$

The bottom surface is in the conductor where $\vec{D}=0$ hence corresponding integral is zero

The top surface is in the free space and we are interested in the boundary condition, hence top surface can be shifted at the boundary with $\Delta h \to 0$.

$$\therefore \int_{top} \vec{D}\cdot\vec{ds} + \int_{lateral} \vec{D}\cdot\vec{ds} = Q \quad \text{---- (2)}$$

The lateral surface area is $2\pi r \Delta h$

where $r \to$ radius of the cylinder

But $\Delta h \to 0$, this area reduces to zero and corresponding integral is zero.

while only component of $\vec{D}$ present is the normal component having magnitude D_N. The top surface is very small over which D_N can be assumed constant and can be taken out of integration.

$$\therefore \int_{top} \vec{D}\cdot\vec{ds} = D_N \int_{top} \vec{ds} = D_N \Delta s \quad \text{--- (3)}$$

$$\therefore D_N \Delta s = Q \quad \text{--- (4)}$$

But at Boundary, ~~condition~~ the charge exists in the form of surface charge density $\rho_s\ C/m^2$

$$\therefore Q = \rho_s \Delta s \quad \text{---- (5)}$$

sub 5 in 4 we get

$$D_N \Delta S = \rho_s \Delta S$$

$$\therefore \boxed{D_N = \rho_s}$$

Thus the flux leaving normally and the normal component of flux density is equal to the surface charge density.

$$\therefore D_N = \varepsilon_0 E_N = \rho_s$$

$$\therefore E_N = \frac{\rho_s}{\varepsilon_0}.$$

BOUNDARY CONDITION B/W CONDUCTOR & DIELECTRIC

The free space is a dielectric with $\varepsilon = \varepsilon_0$. Thus if the boundary is between conductor and dielectric $\varepsilon = \varepsilon_0 \varepsilon_r$.

$$\therefore \boxed{\begin{array}{l} E_{tan} = D_{tan} = 0 \\ D_N = \rho_s \\ E_N = \dfrac{\rho_s}{\varepsilon} = \dfrac{\rho_s}{\varepsilon_0 \varepsilon_r}. \end{array}}$$

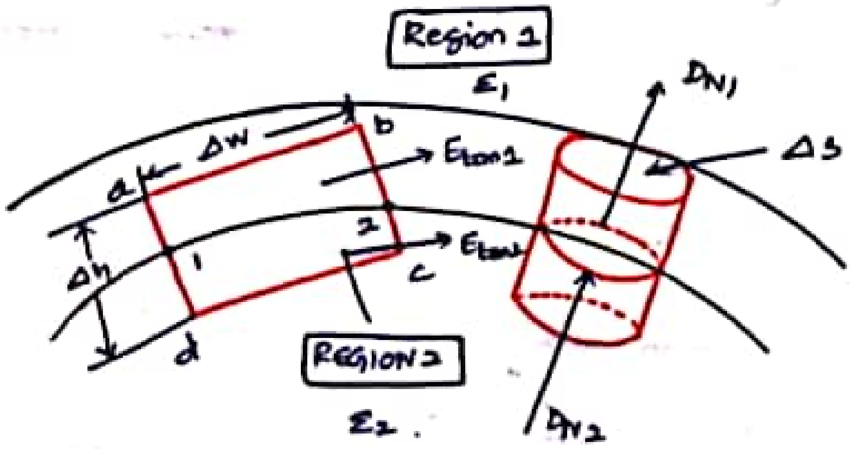
BOUNDARY CONDITIONS BETWEEN TWO PERFECT DIELECTRICS:

Let us consider the boundary b/w two perfect dielectrics. one dielectric has permittivity ε_1 while other has permittivity ε_2. The interface is shown in the figure.

The $\vec{E}$ and $\vec{D}$ are to be obtained again by resolving each into two components, tangential to the boundary and normal to the surface.

Consider a closed loop abcda rectangular in shape having elementary heigh Δh and elementary width Δw, as shown in figure.

It is placed in such a way that $\Delta h/2$ is in the dielectric 1 while the remaining is dielectric 2. Let us evaluate the integral $\vec{E}\cdot\vec{dL}$ along this path, tracing it in clockwise direction as a-b-c-d-a.



$$\oint \vec{E}\cdot\vec{dL} = 0 \quad \text{---(1)}$$

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} + \int_b^c \vec{E}\cdot\vec{dL} + \int_c^d \vec{E}\cdot\vec{dL} + \int_d^a \vec{E}\cdot\vec{dL} = 0 \quad \text{---(2)}$$

Now $\vec{E_1} = \vec{E_{1t}} + \vec{E_{1N}}$

$\vec{E_2} = \vec{E_{2t}} + \vec{E_{2N}}$

Both $\vec{E_1}$ and $\vec{E_2}$ in the respective dielectrics have both the components, normal and tangential.

Let $|\vec{E_{1t}}| = E_{tan1}$ $\quad |\vec{E_{2t}}| = E_{tan2}$

$|\vec{E_{1N}}| = E_{1N}$ $\quad |\vec{E_{2N}}| = E_{2N}$.

Now for the rectangle to be reduced at the surface to analyse boundary conditions, $\Delta h \to 0$

As $\Delta h \to 0$ $\int_b^c$ and $\int_d^a$ become zero as these are line integrals along Δh and $\Delta h \to 0$, Hence eqn (2) becomes

$$\int_a^b \vec{E}\cdot\vec{dL} + \int_c^d \vec{E}\cdot\vec{dL} = 0 \quad \text{---} (3)$$

Now a–b is in dielectric 1 hence the corresponding component of $\vec{E}$ is E_{tan1} as a-b direction is tangential to the surface

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} = E_{tan1} \int_a^b \vec{dL} = E_{tan1}(\Delta W) \quad \text{---} (4)$$

while c-d is in dielectric 2 hence the corresponding component of $\vec{E}$ is E_{tan2} as c-d direction is also tangential to the surface. But the direction of c-d is opposite to a-b hence corresponding integral is negative as the integral obtained for path a-b.

$$\therefore \int_c^d \vec{E}\cdot\vec{dL} = -E_{tan2}(\Delta W) \quad \text{----} (5)$$

substituting (4) and (5) in (3) we get

$$E_{tan1}(\Delta W) - E_{tan2}(\Delta W) = 0$$

$$\Rightarrow \boxed{E_{tan1} = E_{tan2}} \quad \text{---} (6)$$

Thus The tangential component of field intensity at the boundary in both the dielectrics remain same i.e Electric field intensity is continuous across the boundary

The relation b/w $\vec{D}$ and $\vec{E}$ is known as,

$$\vec{D} = \epsilon\vec{E}$$

Hence if D_{tan1} and D_{tan2} are magnitudes of the tangential components of $\vec{D}$ in dielectric 1 and 2 respectively then,

$$D_{tan1} = \varepsilon_1 E_{tan1}$$

$$D_{tan2} = \varepsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{\varepsilon_1} = \frac{D_{tan2}}{\varepsilon_2}$$

$$\boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \quad \text{---- (7)}$$

Thus tangential components of $\vec{D}$ undergoes some change across the interface hence tangential $\vec{D}$ is said to be discontinuous across the boundary.

To find the normal components, let us use Gauss's law. Consider a Gaussian surface in the form of right circular cylinder, placed in such a way that half of it lies in dielectric 1 while the remaining half in dielectric 2. The height $\Delta h \to 0$ hence flux leaving from its lateral surface is zero. The surface area of its top and bottom is ΔS.

$$\therefore \oint \vec{D} \cdot \vec{ds} = Q \quad \text{--- (8)}$$

$$\therefore \left[\int_{top} + \int_{bottom} + \int_{lateral} \right] \vec{D} \cdot \vec{ds} = Q \quad \text{---- (9)}$$

But $\displaystyle\int_{lateral} \vec{D} \cdot \vec{ds} = 0$ as $\Delta h \to 0$. ---- (10)

$$\therefore \int_{top} \vec{D} \cdot \vec{ds} + \int_{\cancel{lateral}\ bottom} \vec{D} \cdot \vec{ds} = Q \ . \quad \text{---- (11)}$$

The flux leaving normal to the boundary is normal to the top and bottom surfaces.

$\therefore |\vec{D}| = D_{N1}$ for dielectric 1.

$= D_{N2}$ for dielectric 2.

As the top and bottom surfaces are elementary, flux density can be assumed. constant and can be taken out as integration

$$\therefore \int_{top} \vec{D} . \vec{ds} = D_{N1} \int_{top} \vec{ds} = D_{N1} \Delta S \quad ---- (12)$$

For top surface, the direction of D_N is entering the boundary while for bottom surface, the direction of D_N is leaving the boundary.

Both are opposite in direction, at the boundary

$$\therefore \int_{bottom} \vec{D} . \vec{ds} = - D_{N2} \int_{bottom} \vec{ds} = - D_{N2} \Delta S \quad --- (13)$$

Sub (12) and (13) in (11) we get

$$D_{N1} \Delta S - D_{N2} \Delta S = Q$$

But $Q = \rho_S \Delta S$

$$\cancel{D_{N1} = \rho_{N2}}$$

$$\therefore \boxed{D_{N1} - D_{N2} = \rho_S}$$

There is no free charge avaliable in. Perfect dielectric and hence no free charge can exist on the surface. All charges in dielectric are bound charges and are not free.

Hence at ideal dielectric media boundary the surface charge density ρ_s can be assumed zero.

$\therefore \rho_s = 0$

$D_{N1} - D_{N2} = 0$

$$\boxed{D_{N1} = D_{N2}}$$

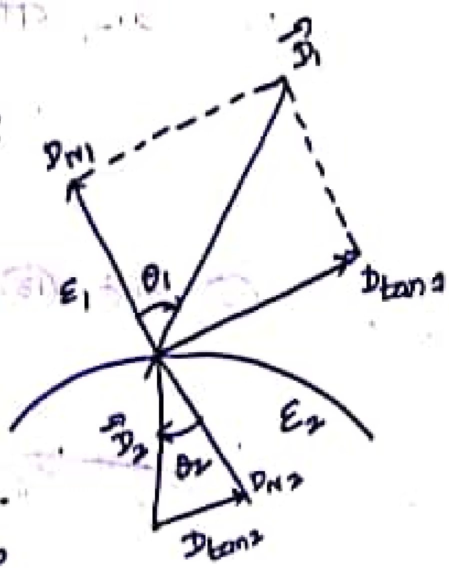
Hence the normal component of flux density $\vec{D}$ is continuous at the boundary b/w the two perfect dielectrics.

$\therefore D_{N1} = \varepsilon_1 E_{N1}$ and $D_{N2} = \varepsilon_2 E_{N2}$

$$\therefore \frac{D_{N1}}{D_{N2}} = \frac{\varepsilon_1}{\varepsilon_2} \frac{E_{N1}}{E_{N2}} = 1.$$

$$\boxed{\therefore \frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}.}$$

Refraction of $\vec{D}$ at the Boundary:



The directions of $\vec{D}$ and $\vec{E}$ change at the boundary b/w the two dielectrics.

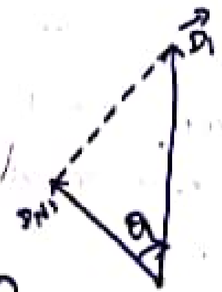
Let $\vec{D_1}$ and $\vec{E_1}$ make an angle θ_1 with the normal to the surface. $\vec{D_1}$ and $\vec{E_1}$ direction is same as

$$\boxed{\vec{D_1} = \varepsilon_1 \vec{E_1}}$$

This is shown in the figure.

Let $|\vec{D_1}| = D_1$

$|\vec{D_2}| = D_2$



from fig

$\cos\theta_1 = \frac{D_{N1}}{D_1}$

$\therefore D_{N1} = D_1 \cos\theta_1$ --- (1)

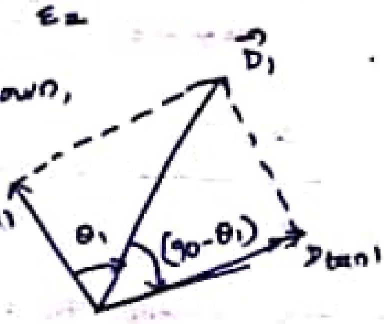
$|||^{ly}$ $D_{N2} = D_2 \cos\theta_2$ --- (2)

But W.K.T $D_{N1} = D_{N2}$

$\therefore D_1 \cos\theta_1 = D_2 \cos\theta_2$.

$|||^{ly}$ W.K.T $\dfrac{D_{tan1}}{D_{tan2}} = \dfrac{\epsilon_1}{\epsilon_2}$

From the Figure shown,



$\cos(90-\theta_1) = \dfrac{D_{tan1}}{D_1}$

$\sin\theta_1 = \dfrac{D_{tan1}}{D_1}$

$$\left.\begin{aligned} \Rightarrow D_{tan1} &= D_1 \sin\theta_1 \\ |||^{ly} \quad D_{tan2} &= D_2 \sin\theta_2 \end{aligned}\right\} \text{--- (3)}$$

$$\frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{D_{tan1}}{D_{tan2}} \text{ --- (A)}$$

Now (3a) $\div$ (1)

$$\frac{D_1 \sin\theta_1}{D_1 \cos\theta_1} = \frac{D_{tan1}}{D_{N1}} \Rightarrow \tan\theta_1 = \frac{D_{tan1}}{D_{N1}} \text{ --- (4a)}$$

$|||^{ly}$ $\tan\theta_2 = \dfrac{D_{tan2}}{D_{N2}}$ --- (4b)

$$\frac{(4a)}{(4b)} \Rightarrow \frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} \cdot \frac{D_{N2}}{D_{N1}}$$

But $D_{N1} = D_{N2}$

$$\therefore \frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} = \frac{\epsilon_1}{\epsilon_2}$$

$$\Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}}$$

This is called law of refraction. Thus the angles θ_1 and θ_2 are dependent on permittivities of two media and not on $\vec{D}$ or $\vec{E}$.

If $\epsilon_1 > \epsilon_2$ then $\theta_1 > \theta_2$

The magnitude of $\vec{D}$ in region 2 can be obtained as

$$D_2^2 = D_{N2}^2 + D_{tan2}^2 = (D_1 \cos\theta_1)^2 + D_{tan2}^2$$

Now $D_{tan2} = D_2 \sin\theta_2 = \dfrac{D_1 \sin\theta_1 \epsilon_2}{\epsilon_1}$. [from (A)]

$\therefore$ $D_2^2 = (D_1\cos\theta_1)^2 + \left(D_1 \sin\theta_1 \dfrac{\epsilon_2}{\epsilon_1}\right)^2$

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}.$$

III^{rly} Magnitude of E_2 can be obtained as

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1}$$

The equations shows that

i) $\vec{D}$ is larger in region of larger permittivity

ii) $\vec{E}$ is larger in region of smaller permittivity

iii) $|\vec{D_1}| = |\vec{D_2}|$ if $\theta_1 = \theta_2 = 0^\circ$

(iv) $|\vec{E_1}| = |\vec{E_2}|$ if $\theta_1 = \theta_2 = 90^\circ$.

To find the angles θ_1 and θ_2, w.r.to normal use the dot product if normal direction to the boundary is known.

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