21.11 *LC* NETWORK SYNTHESIS

Any *LC* network can be synthesized in two forms designated by the names of the inventors *Foster and Cauer.*

21.11.1 Foster's Canonic Form

A positive real impedance Z (s) of LC network being a ratio of either even to odd or odd to even polynomials, it can be synthesized by *Foster form. Since the synthesis is performed with the least number of elements hence the synthesis gives birth to a network with least number of LC elements and termed as cenomic form of network.*

In Foster synthesis there are two types of networks for realisation of one port reactive networks. The first type is a series combination of parallel *LC* networks while the other is the parallel combination of series LC networks. (Figs. 21.14 and 21.15)

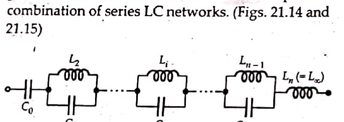

Fig. 21.14 First form of Foster *LC* network (Impedance Form)

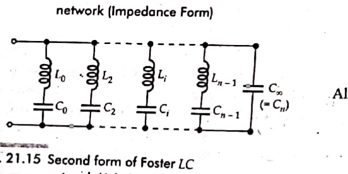

Fig. 21.15 Second form of Foster *LC* network (Admittance Form)

Z(s) is a *PR* function, having simple poles at $j\omega$ axis $[s = \pm j\omega_i, i = 1, 2, \ldots, m]$ as well as simple poles at $s = 0$ and $s = \infty$, using partial fraction, *Z(s)* can be expressed as

$$Z(s) = \frac{A_0}{s} + \sum_{i=2}^{n-1} \frac{2A_i s}{s^2 + \omega_i^2} + H.s \quad \ldots(21.31)$$

when the constituent terms in *Z(s)* can be interpreted as

(*a*) $\frac{A_0}{s}$ results from a possible pole at $s = 0$ [$\frac{A_0}{s}$ is the first term]

(*b*) *H.s* results from a possible pole at $s = \infty$ [*H.s* is the last (*n*th) term]

(*c*) $\frac{2A_i s}{s^2 + \omega_i^2}$ results from a pair of conjugate poles on the $j\omega$ axis and obviously,

$$\frac{2A_i s}{s^2 + \omega_i^2} = \frac{A_i}{s + j\omega_i} + \frac{A_i'}{s - j\omega_i} = \frac{(A_i + A_i')s - j(A_i - A_i')\omega_i}{s^2 + \omega_i^2} \quad \ldots(21.32)$$

In order that they give a real term after combination, it is essential that the residues A_i and A_i' must be equal.

Following Fig. 21.14, for the parallel combination of *L, C,*

$$Z(s) = \frac{1}{Cs + \frac{1}{Ls}}$$

i.e.,

$$Z(s) = \frac{\left(\frac{1}{C}\right) s}{s^2 + \frac{1}{LC}} \quad \ldots(21.33)$$

However, from equation (21.31),

$$Z(s) = \frac{A_0}{s} + \frac{2A_2 s}{s^2 + \omega_2^2} + \ldots + \frac{2A_{n-1} s}{s^2 + \omega_{n-1}^2} + H.s \quad \ldots(21.34)$$

Also, $Z(s) = Z_1(s) + Z_2(s) + \ldots + Z_n(s) \quad \ldots(21.35)$

Comparing equations (21.34) and (21.35),

$$Z_1(s) = \frac{A_0}{s} \quad \ldots(21.36)$$

$$Z_n(s) = H.s \quad \ldots(21.37)$$

It is evident that equation (21.36) represents that Z_1 (s) *is an impedance representing a capacitor* C_0 *of value* $(1/A_0)$ and from equation (21.37), it is evident that Z_n (s) *represents an inductor of inductance H henry* (L_∞). *The remaining intermediate terms represent parallel combination of an inductor and a capacitor.* Comparing Eq. (21.33) with the mid-terms of equation (21.31), we get,

$$C_i = \frac{1}{2A_i} \text{ and } L_i = \frac{2A_i}{\omega_i^2} \quad \ldots(21.38)$$

[*i*th term refers to $(2A_i\, s/s^2 + \omega_i^2)$ in equation (21.31)].

The above realisation is known as First *Form of Foster Realisation of LC network.*

The *second form of Foster network is a parallel combination of series LC networks* (Fig. 21.15). As all the series branches are connected in parallel, it is convenient to consider the driving point admittance (Y (s)).

If we write the expression for LC series combination,

$$Y_{series}(s) = \frac{1}{Ls + \frac{1}{Cs}} = \frac{(s/L)}{s^2 + \frac{1}{LC}} \quad \ldots(21.39)$$

However equation (21.31) can be written as

$$Y(s) = \frac{B_0}{s} + \frac{2B_2 s}{s^2 + \omega_2^2} + \ldots + \frac{2B_i s}{s^2 + \omega_i^2} + \ldots + \frac{2B_{n-1} s}{s^2 + \omega_{n-1}^2} + H.s \quad \ldots(21.40)$$

Also, if we take the total admittance into a parallel combination of branch admittances,

$$Y(s) = Y_1(s) + Y_2(s) + \ldots + Y_n(s) \quad \ldots(21.41)$$

Comparing equations (21.40) and (21.41), $Y_1(s) = (B_0/s)$ *represent an inductor* L_0 *of value* $(1/B_0)$ *while* $Y_n(s) = Hs$*, represents a capacitor* C_∞ *of value H Farad. The intermediate terms represent series combination of an inductor and capacitor.* By comparing eqns. (21.39) and (21.40), we have

$$L_i = \frac{1}{2B_i}; C_i = \frac{2B_i}{\omega_i^2}$$

[*n* refers to the term $(2B_n s/s^2 + \omega_n^2)$ in eqn. (21.40)].

21.12 SIGNIFICANCE OF ELEMENTS IN THE FOSTER FORM

First Foster Form

(*a*) The capacitor C_0 represents the term $\left(\frac{A_0}{s}\right)$ that corresponds to a pole at the origin.

(*b*) The inductor L_∞ represents the term *Hs* that corresponds to a pole at the infinity.

(*c*) The parallel $L_i C_i$ networks represent the term $(s^2 + \omega_i^2)$ in the denominator of $Z(s)$. ω_i, the frequency represents a pole frequency. Parallel resonance of L_i and C_i would result in a pole of Z(s). Reactance of each parallel circuit changes sign as frequency increases and at some frequency the reactance of a certain group of these parallel circuits is equal and opposite to the reactance of all remaining parallel circuits. This results in a zero of *Z(s)*.

(*d*) When there is no *s* term in the denominator of *Z(s)*, there will be no $\left(\frac{A_0}{s}\right)$ term in the partial fraction expansion of Z(s). C_0 is absent and replaced by short circuit.

Again, when the order of the numerator polynomial is less than that of the denominator polynomial, there will be no *Hs* factor in the partial expansion fraction of Z(s) and hence there would be no inductor L_0 in the circuit. L_0 will then be replaced by a short circuit.

Second Foster Form

(*a*) The inductor L_0 represents the term $\left(\frac{B_0}{s}\right)$ in the partial fraction form of Y(s) corresponding a pole at the origin.

(*b*) The capacitor C_∞ represents the term (*Hs*) corresponding a pole at infinity.

(*c*) The series $L_i C_i$ network represents the term $(s^2 + \omega_i^2)$ in the denominator of Y(s). Frequency ω_n represents a pole frequency that is associated with series resonance of the LC network. Series resonance within these LC combinations causes the network to have a pole of admittance. Susceptance of series circuit changes sign as frequency increases. At some frequency the susceptance of a certain group of these series circuits is equal and opposite to the remaining series circuits. This yields a zero of Y(s).

(*d*) In case there is no pole either at $\omega = 0$ or at $\omega = \infty$, or at both, the presence of zero is signified with absence of end elements. The second form of Foster network can then be realised by the absence of C_0 and L_∞, the end elements.

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To conclude with the discussion about the two forms of Foster networks, it may be said that the first form of realisation is actually the series impedance realisation of the driving point impedance $Z(s)$. The second form is a parallel admittance realisation of driving point admittance $Y(s)$. $Y(s)$ can be obtained by inversion of $Z(s)$. However, this process of inversion interchanges poles and zeros.

To summarise, we thus can say that if there is a pole at $\omega = 0$, the first element C_0 is present in the network of first Foster form. Similarly, if there is a pole at $\omega = \infty$, the last element L_∞ is present in it. In the second form of network, if there is a pole at $\omega = 0$, the first element L_0 is present while for $\omega = \infty$, the last element C_∞ is present.

EXAMPLE 21.35 *The driving point impedance of an LC network is given by*

$$Z(s) = 10\frac{(s^2+4)(s^2+16)}{s(s^2+9)}$$

Obtain the first form of Foster network.

SOLUTION. Observation reveals that there is s^4 term in the numerator while s^3 term in denominator. Since the numerator contains an excess term, while the denominator has one s term, hence two poles exist at $\omega=0$ and at $\omega=\infty$. Thus, the network consists of first and last element in the Foster first form.

By taking the partial fraction expansion of $Z(s)$, we find

$$Z(s) = \frac{A_0}{s} + \frac{A_2}{s+j3} + \frac{A_2^*}{s-j3} + Hs$$

where $$A_0 = \left.\frac{10(s^2+4)(s^2+16)}{(s^2+9)}\right|_{s=0}$$

$$= \frac{10\times 4\times 16}{9} = 71.11$$

$$A_2 = \left.\frac{10(s^2+4)(s^2+16)}{s(s-j3)}\right|_{s=-j3}$$

$$= \frac{10[\{(-j3)^2+4\}\{(-j3)^2+16\}]}{(-j3)(-j3-j3)}$$

$$= 10\left[\frac{(-9+4)(-9+16)}{-18}\right]$$

$$= \frac{350}{18} = 19.45$$

$$\therefore \quad C_0 = \frac{1}{A_0} = \frac{1}{71.11}\ \text{F} = 0.0141\ \text{F}$$

$$L_\infty = H = 10\ \text{H}$$

By inspection, the constant multiplier term is H i.e., $H=10$

$$C_2 = \frac{1}{2A_2} = \frac{1}{2\times 19.45} = 0.0257\ \text{F}$$

$$L_2 = \frac{2A_2}{\omega_n^2} = \frac{2\times 19.45}{3^2}$$

[$\omega_n = 3$ since the denominator contains s^2+3^2 in the form of $s^2+\omega_n^2$]

$$= 4.322\ \text{H}.$$

The first form of Foster network is shown in Fig. E21.11.

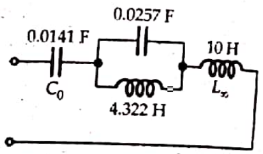

Fig. E21.11

EXAMPLE 21.36 *Find the second Foster Form of the admittance function :*

$$Y(s) = \frac{s(s^2+9)}{10(s^2+4)(s^2+25)}.$$

SOLUTION. Since there is an s term in the numerator and an excess term in the denominator than in the numerator, two zeros exist at $\omega=0$ and $\omega=\infty$. Thus, there is no question of first and last element and the given network consists of LC combinations of parallel elements.

$$Y(s) = \frac{2B_1 s}{s^2+4} + \frac{2B_2 s}{s^2+25}$$

$$B_1 = \left.\frac{1}{10}\frac{s(s^2+9)}{(s-j2)(s^2+25)}\right|_{s=-j2}$$

$$= \frac{1}{10}\frac{-j2(-4+9)}{(-j4)(-4+25)} = \frac{1}{10}\cdot\frac{-10}{-84} = \frac{1}{84}$$

$$B_2 = \left.\frac{1}{10}\frac{s(s^2+9)}{(s^2+4)(s-j5)}\right|_{s=-j5}$$

$$= \frac{1}{10}\frac{(-j5)(-25+9)}{(-25+4)(-j10)} = \frac{1}{10}\frac{80}{210} = \frac{1}{26.25}$$

$\therefore$ The values of elements in the 2nd form of Foster network,

$$L_1 = \frac{1}{2B_1} = 42\ \text{H}$$

$$C_1 = \frac{2B_1}{\omega_1^2} = \frac{2\times 1/84}{2^2} = \frac{1}{168}\ \text{F}$$

$$L_2 = \frac{1}{2B_2} = 26.25/2 = 13.125\ \text{H}$$

$$C_2 = \frac{2B_2}{\omega_2^2} = \frac{2\times 1/26.25}{5^2} = \frac{2}{656.25}\ \text{F}.$$

Figure E21.12 represents the second form of Foster network.

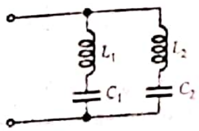

Fig. E21.12

EXAMPLE 21.37 *The driving point impedance of a one port LC network is given by*

$$Z(s) = \frac{2(s^2+9)(s^2+16)}{s(s^2+4)}$$

Obtain the Foster form of LC network realisation.

SOLUTION. Observation of the expression of $Z(s)$ reveals that there is a s^4 term in the numerator while there is only s^3 term in the denominator. Since the numerator has one excess term while the denominator has one s term, hence two poles exist at $\omega=0$ and at $\omega=\infty$. The given network thus consists of first and last elements in the first form of Foster network.

The partial fraction expansion gives

$$Z(s) = \frac{A_0}{s} + \frac{A_2}{s+j2} + \frac{A_2^*}{s-j2} + Hs$$

where, $$A_0 = \left.\frac{2(s^2+9)(s^2+16)}{s^2+4}\right|_{s=0}$$

$$= \frac{2\times 9\times 16}{4} = 72$$

$$A_2 = \left.\frac{2(s^2+9)(s^2+16)}{s(s-j2)}\right|_{s=-j2}$$

$$= \frac{2(-4+9)(-4+16)}{(-j2)(-j4)}$$

$$= \frac{2\times 5\times (12)}{-8} = -15$$

Since A_2 has come out to be –ve hence Foster realisation is not possible.

As per Foster realisation, the coefficients of the partial fraction expansion must be real and +ve. Also the poles and zeros of $Z(s)$ do not interlace and hence $Z(s)$ is not basically an LC function.

EXAMPLE 21.38 *The driving point impedance of a one port LC network is given by*

$$Z(s) = \frac{8(s^2+4)(s^2+25)}{s(s^2+16)}$$

Obtain the first and second Foster form of equivalent networks.

SOLUTION. Since the order of the numerator polynomial exceeds that of the denominator polynomial hence there is a simple pole at $\omega=\infty$. On the other hand, the lowest order term of the numerator is of order lower than that of the denominator. Hence there is a simple pole at $\omega=0$. This indicates that both first and last elements would be present.

Thus, the partial fraction expansion gives

$$Z(s) = \frac{A_0}{s} + \frac{A_2}{s+j4} + \frac{A_2^*}{s-j4} + Hs$$

Here, $$A_0 = \left.\frac{8(s^2+4)(s^2+25)}{s^2+16}\right|_{s=0}$$

$$= \frac{8\times 4\times 25}{16} = 50$$

$$A_2 = \left.\frac{8(s^2+4)(s^2+25)}{s(s-j4)}\right|_{s=-j4}$$

$$= \frac{8(-16+4)(-16+25)}{-j4\times -j8}$$

$$= \frac{8\times(-12)\times 9}{-32} = 27$$

By inspection, $H = 8$.

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∴ In the first form of Foster network,

$$C_0 = \frac{1}{A_0} = \frac{1}{50} \text{ Farad}$$

$$L_\infty = H = 8 \text{ Henry}$$

$$C_2 = \frac{1}{2A_2} = \frac{1}{54} \text{ Fared}$$

$$L_2 = \frac{2A_2}{\omega_n^2} = \frac{54}{4^2} = 3.375 \text{ Henry.}$$

Figure E21.13 represents the First form of Foster network.

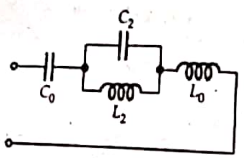

Fig. E21.13

In order to find the second Foster form, we shall have to represent the given function into admittance form.

$$Y(s) = \frac{s(s^2+16)}{8(s^2+4)(s^2+25)}$$

Since there is one s term in the numerator and an excess term in the denominator than in the numerator, two zeros exist at $\omega = 0$ and at $\omega = \infty$. Thus, the given network is only a LC parallel combination. The presence of zero indicates that there would be no end elements in the 2nd form of Foster Network. The partial fraction expansion is then

$$Y(s) = \frac{2B_1 s}{s^2+4} + \frac{2B_2 s}{s^2+25}$$

$$B_1 = \frac{1}{8} \cdot \frac{s(s^2+16)}{(s-j2)(s^2+25)}\bigg|_{s=-j2}$$

$$= \frac{-j2 \times 12}{8(-j4 \times 21)} = \frac{24}{84 \times 8} = 0.0357$$

$$B_2 = \frac{1}{8} \frac{s(s^2+16)}{(s^2+4)(s-j5)}\bigg|_{s=-j5}$$

$$= \frac{1}{8} \frac{-j5 \times (-9)}{-21 \times -j10} = 0.027$$

∴ The values of elements in the 2nd form are then

$$L_1 = \frac{1}{2B_1} = \frac{1}{0.0714} \text{ Henry}$$

$$C_1 = \frac{2B_1}{\omega_1^2} = \frac{2 \times 0.0357}{2^2} = 0.0178 \text{ F}$$

$$L_2 = \frac{1}{2B_2} = \frac{1}{2 \times 0.027} = 18.52 \text{ Henry}$$

$$C_2 = \frac{2B_2}{\omega_2^2} = \frac{2 \times 0.027}{5^2} = 0.00216 \text{ Farad}$$

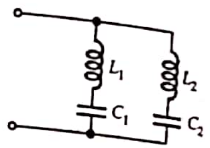

Fig. E21.14

Fig. E21.14 represents the second form of Foster network.

EXAMPLE 21.39 *The driving point impedance of a one port reactive network is given by*

$$Z(s) = 4\frac{s(s^2+4)}{(s^2+1)(s^2+16)}$$

Obtain the Foster forms of LC network realisation.

SOLUTION. Observation reveals that the order of the numerator is less than that of the denominator [in numerator the order is s^3 while in denominator it is s^4]. Hence there is a simple zero at $\omega = \infty$. Further, the lowest ordered term in the numerator is of higher order than the lowest order term of the denominator. Hence there is a simple zero at $\omega = 0$. This indicates that in the first form of the Foster network, there will be no end elements. *i.e.*, there will be no (A_0/s) term and (Hs) terms in the partial fraction expansion.

The partial fraction of $Z(s)$ becomes,

$$Z(s) = \frac{2A_1 s}{s^2+1} + \frac{2A_2 s}{s^2+4^2}.$$

Here $$A_1 = \frac{4s(s^2+4)}{(s-j1)(s^2+16)}\bigg|_{s=-j1}$$

$$= \frac{4(-j1)(-1+4)}{-j2(-1+16)} = \frac{12}{30} = \frac{2}{5} = 0.4$$

$$A_2 = \frac{4s(s^2+4)}{(s^2+1)(s-j4)}\bigg|_{s=-j4}$$

$$= \frac{4(-j4)(-16+4)}{(-16+1)(-j8)}$$

$$= \frac{4 \times (-j4)(-12)}{(-15)(-j8)} = 1.6.$$

∴ In Foster 1st form,

$$C_1 = \frac{1}{2A_1} = \frac{1}{0.8} \text{ F}$$

$$L_1 = \frac{2A_1}{\omega_1^2} = \frac{2 \times 0.4}{1^2} = 0.8 \text{ H}$$

$$C_2 = \frac{1}{2A_2} = \frac{1}{2 \times 1.6} = \frac{1}{3.2} \text{ F}$$

$$L_2 = \frac{2A_2}{\omega_2^2} = \frac{2 \times 1.6}{4^2} = 0.2 \text{ H.}$$

Figure E21.15 represents Foster 1st form of network.

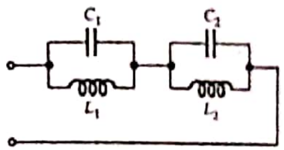

Fig. E21.15

In order to determine the second Foster form, first we consider the admittance function $Y(s)$.

$$Y(s) = \frac{1}{Z(s)} = \frac{(s^2+1)(s^2+16)}{4s(s^2+4)}$$

The numerator polynomial is of order s^4 and the denominator polynomial is of order s^3. The numerator has one excess term and the denominator has s term. Thus the given function has two poles at $\omega = 0$ and $\omega = \infty$. The given network would then contain the end elements in the Foster first form.

The partial fraction expansion gives

$$Y(s) = \frac{B_0}{s} + \frac{B_2}{s+j2} + \frac{B_2^*}{s-j2} + Hs$$

$$B_0 = \frac{(s^2+1)(s^2+16)}{4(s^2+4)}\bigg|_{s=0} = 1$$

$$B_2 = \frac{(s^2+1)(s^2+16)}{4s(s-j2)}\bigg|_{s=-j2}$$

$$= \frac{(-4+1)(-4+16)}{4 \times (-j2)(-j4)}$$

$$= \frac{-3 \times 12}{-32} = 1.125$$

By observation of $Y(s)$, $H = \frac{1}{4} = 0.25$

∴ $$L_0 = \frac{1}{B_0} = 1 \text{ H}$$

$$C_\infty = H = 0.25 \text{ F}$$

$$L_2 = \frac{1}{2B_2} = \frac{1}{2 \times 1.125} = \frac{1}{2.25} \text{ H}$$

$$C_2 = \frac{2B_2}{\omega_2^2} = \frac{2 \times 1.125}{2^2} = 0.5625 \text{ F}$$

Figure E21.16 represents the second form of Foster network.

Fig. E21.16

EXAMPLE 21.40 *Find the first and second Foster form of the driving point impedance function*

$$Z(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)}$$

SOLUTION. Since the order of the numerator polynomial exceeds that of the denominator polynomial hence there is a simple pole at $\omega = \infty$. On the other hand, the lowest order term of the numerator is of the order lower than that of the denominator. Hence there is a simple pole at $\omega = 0$. Thus there would be first and last elements present in the first form of Foster Network.

Thus, the partial expansion gives

$$Z(s) = \frac{A_0}{s} + \frac{A_2}{s+j2} + \frac{A_2^*}{s-j2} + Hs$$

where, $$A_0 = \frac{2(s^2+1)(s^2+9)}{s^2+4}\bigg|_{s=0} = \frac{2 \times 9}{4} = 4.5$$

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$$A_2 = \left.\frac{2(s^2+1)(s^2+9)}{s(s-j2)}\right|_{s=-j2}$$

$$= \frac{2(-4+1)(-4+9)}{(-j2)\times(-j4)}$$

$$= \frac{-3\times 2\times 5}{-8} = 3.75$$

By inspection, $H = 2$

∴ In the first form of Foster network,

$$C_0 = \frac{1}{A_0} = \frac{1}{4.5}\text{ F}$$

$$L_\infty = H = 2\text{ Henry}$$

$$C_2 = \frac{1}{2A_2} = \frac{1}{2\times 3.75} = \frac{1}{7.5}\text{ F}$$

$$L_2 = \frac{2A_2}{\omega_2^2} = \frac{2\times 3.75}{2^2} = 1.875\text{ H}$$

Figure E21.17 represents the 1st form of Foster network.

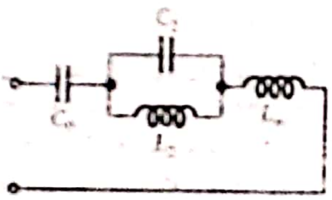

Fig. E21.17

In order to find the second Foster form, we will represent the given function into admittance form.

$$Y(s) = \frac{s(s^2+4)}{2(s^2+1)(s^2+9)}.$$

Since there is one s term in the numerator and an excess term in the denominator than in the numerator, two zeros exist at $\omega = 0$ and at $\omega = \infty$. Thus, the given network is simply a LC parallel combination. The presence of zero indicates that there would be no end elements in the 2nd form of network.

The partial fraction expansion is then

$$Y(s) = \frac{2B_1 s}{s^2+1} + \frac{2B_2 s}{s^2+25}$$

where

$$B_1 = \left.\frac{1}{2}\cdot\frac{s(s^2+4)}{(s-j1)(s^2+9)}\right|_{s=-j1}$$

$$= \frac{1}{2}\frac{(-j1)(-1+4)}{(-j2)(-1+9)} = \frac{3}{32}$$

$$B_2 = \left.\frac{1}{2}\cdot\frac{s(s^2+4)}{(s^2+1)(s-j3)}\right|_{s=-j3}$$

$$= \frac{1}{2}\frac{(-j3)(-9+4)}{(-9+1)(-j6)}$$

$$= \frac{15}{48\times 2} = \frac{15}{96} = \frac{5}{32}.$$

The values of elements in the second form are thus

$$L_1 = \frac{1}{2B_1} = \frac{16}{3}\text{ H}$$

$$L_2 = \frac{1}{2B_2} = 3.2\text{ H}$$

$$C_1 = \frac{2B_1}{\omega_1^2} = \frac{2\times 3/32}{1} = \frac{3}{16}\text{ F}$$

$$C_2 = \frac{2B_2}{\omega_2^2} = \frac{2\times 5/32}{5^2} = \frac{2\times 5}{32\times 5^2} = \frac{1}{80}\text{ F.}$$

Foster's second form of network is shown in Fig. E21.18.

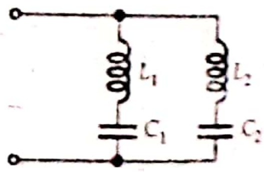

Fig. E21.18

21.13 CAUER CANONIC FORM OF REACTIVE NETWORKS

W. Cauer, in 1927 introduced realisation of one port LC networks in two configurations (commonly known as *Cauer-I and Cauer-II* forms) using the

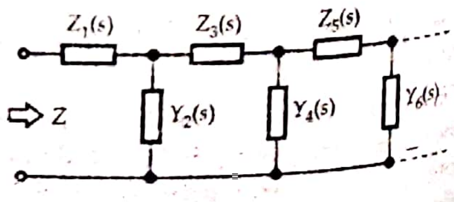

Fig. 21.16 Ladder Network