where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the quotients in each step of the division

$$\frac{M(s)}{N(s)} \text{ or } \frac{N(s)}{M(s)}.$$

These coefficients must be *positive* when the polynomial $P(s)$ is Hurwitz.

(*d*) If the polynomial satisfies the condition of *Hurwitz*, the polynomial then is Hurwitz to an even multiplicative factor $W(s)$; *i.e.*, if $P_1(s) = W(s)P(s)$ and $P(s)$ is Hurwitz, $P_1(s)$ must be Hurwitz.

(*e*) If the ratio of the polynomial $P(s)$ and its derivative $(P'(s))$ gives a continued fraction expansion with all positive coefficients, then $P(s)$ is Hurwitz. In case the polynomial is either *only even* or *only odd*, the method of continued fraction is *not possible* and this method is then beneficial.

The definition of the Hurwitz polynomial presented in this text was developed by Brune and later illustrated by Van Valkenburg and Kuo. This definition refers the criteria of the polynomial to be Hurwitz bounded by characterisation of poles of the system. Guillemin as well as Van Valkenburg had also bounded the criteria of polynomial to be Hurwitz by characterisation of zeros of the polynomial as mentioned below :

The characterisation of Hurwitz poynomial by zeros states that a real polynomial $P(s)$ [*i.e.*, polynomial with real coefficients] is termed as a Hurwitz if all its zeros lie in the left half plane (*i.e.*, $\text{Re}(s) \le 0$) with those on the imaginary axis being simple ($P(s)$ is said to be strictly Hurwitz provided $j\omega$ axis zeros are excluded *i.e.*, all the zeros occur in the left half side of S plane only ($\text{Re}(s) < 0$)). It may be noted that both these approaches result in identical concept as it is evident that if system function being represented by a ratio of polynomials and the denominator being analysed for its poles in order to determine the stability of the system, it is same thing if the zeros of the numerator are analysed for stability, *i.e.*, the analysis of pole of a system is identical to analysing the zeros of the polynomial in of the system function.

21.4 PROCEDURE OF TESTING OF A GIVEN POLYNOMIAL FOR HURWITZ CHARACTER

Physical Testing

The necessary but *not* sufficient conditions for $P(s)$ to be Hurwitz are as follows :

1. All the coefficients of the polynomial must be positive and real.
2. There must not be any power of s missing between the highest degree and lowest degree of the polynomial (unless the polynomial is completely even or completely odd).

Analytic Testing

The necessary and sufficient condition for $P(s)$ to be Hurwitz is as follows :

The quotients $(\alpha_1, \alpha_2 \ldots)$ in the continued fraction expansion of $Z(s)\left[= \frac{M(s)}{N(s)}\right.$ where $M(s)$ represents the even part of $P(s)$ and $N(s)$ the odd part] *must be real and positive.* However if due to a common factor between $M(s)$ and $N(s)$, if the continued fraction method is prematurely terminated, then the quotients in the continued fraction expansion of $\psi(s)\left[\psi(s) = \frac{P(s)}{P'(s)}, P'(s)\right.$ being the first derivative of $P(s)\Big]$ must be all real and positive.

Use of $\psi(s)$ is also suitable if the given polynomial is either only even or only odd.

EXAMPLE 21.2 *Check whether the polynomial*

$$s^5 + 9s^4 + 7s^3 + s^2 + 4s$$

is Hurwitz or not.

SOLUTION.

$$P(s) = s^5 + 9s^4 + 7s^3 + s^2 + 4s$$

$$= M(s) + N(s)$$

where $M(s) = 9s^4 + s^2 = s^2(1 + 9s^2)$

$N(s) = s^5 + 7s^3 + 4s = s(4 + 7s^2 + s^4)$

Let $Z(s) = \frac{N(s)}{M(s)} = \frac{s^5 + 7s^3 + 4s}{9s^4 + s^2}$

$$= \frac{s^4 + 7s^2 + 4}{s(9s^2 + 1)} [s \neq 0].$$

It is obvious that the above function $Z(s)$ has a zero at the origin. The quotient of the continued function of $N(s)/M(s)$ also gives negative value. This proves that the given polynomial is not Hurwitz.

EXAMPLE 21.3 *Check whether the polynomial* $s^8 + s^7 + s^6 - 2s^5 + 4s^3 - s^2 + s + 1$ *is Hurwitz or not.*

SOLUTION. By inspection it is evident that the given polynomial is not Hurwitz because the coefficients of the s^5 and s^2 term are negative.

EXAMPLE 21.4 *Indicate, with proper reasoning which of the following polynomials are Hurwitz :*

(*a*) $s^2 + 4s + 10$

(*b*) $s^4 + s^3 + 2s^2 + 3s + 2$

(*c*) $s^4 + 11s^3 + 39s^2 + 51s + 20$

SOLUTION. (*a*) $P(s) = s^2 + 4s + 10$

The given polynomial is Hurwitz because it is a quadratic, with no missing term and all its coefficients are of positive sign.

(*b*) $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$

$M(s) = s^4 + 2s^2 + 2$

$N(s) = s^3 + 3s$

$$\therefore \quad Z(s) = \frac{M(s)}{N(s)} = \frac{s^4 + 2s^2 + 2}{s^3 + 3s}$$

Using continued fraction method, the function $Z(s) = \frac{M(s)}{N(s)}$ is tested as follows :

$$s^3 + 3s)\, s^4 + 2s^2 + 2\, (s$$
$$\underline{s^4 + 3s^2}$$
$$-s^2 + 2)\, s^3 + 3s\, (-s$$
$$\underline{s^3 - 2s}$$
$$5s)\, -s^2 + 2\, (-\frac{s}{5}$$
$$\underline{-s^2}$$
$$2)\, 5s\, (\frac{5s}{2}$$
$$\underline{5s}$$
$$\times$$

Since the continued fraction contains negative quotients hence the given polynomial is not Hurwitz.

(*c*) $P(s) = s^4 + 11s^3 + 39s^2 + 51s + 20$

$= M(s) + N(s)$

$= \underbrace{(s^4 + 39s^2 + 20)}_{M(s)} + \underbrace{(11s^3 + 51s)}_{N(s)}$

Let us now use the continued fraction method as shown below :

$$11s^3 + 51s)\, s^4 + 39s^2 + 20\, (\frac{s}{11}$$
$$\underline{s^4 + \frac{51}{11}s^2}$$
$$\frac{378}{11}s^2 + 20)\, 11s^3 + 51s\, (\frac{121}{378}s$$
$$\underline{11s^3 + \frac{1210}{189}s}$$
$$\frac{8429}{189}s)\, \frac{378}{11}s^2 + 20\, (\frac{378 \times 189}{8429 \times 11}s$$
$$\underline{\frac{378}{11}s^2}$$
$$20)\, \frac{8429}{189}s\, (\frac{8429}{189 \times 20}s$$
$$\underline{\frac{8429}{189}s}$$
$$\times$$

Since the continued fraction has all +ve quotients hence the given polynomial is Hurwitz.

EXAMPLE 21.5 *Determine whether the given polynomial* $(s + 2)(s^2 + 4s + 6)(s^2 + 3s + 2)$ *is Hurwitz or not.*

SOLUTION. The polynomial is given in form of factors. No further factorisation is needed. Also all the factors are Hurwitz. Thus the given polynomial is Hurwitz.

EXAMPLE 21.6 *Check whether the following polynomials are Hurwitz or not :*

(*a*) $s^4 + 7s^3 + 4s^2 + 18s + 6$

(*b*) $s^4 + s^3 + 6s^2 + 3s + 6$

(*c*) $s^4 + s^3 + 2s^2 + 4s + 1$

SOLUTION. (*a*)

$$P(s) = s^4 + 7s^3 + 4s^2 + 18s + 6$$

$$= M(s) + N(s)$$

where $M(s) = s^4 + 4s^2 + 6$

and $N(s) = 7s^3 + 18s$

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Using continued fraction method, from

$$Z(s)=\frac{M(s)}{N(s)},$$

$$7s^3+18s)\ s^4+4s^2+6\ (\tfrac{s}{7}$$
$$s^4+\frac{18s^2}{7}$$
$$\frac{10s^2}{7}+6)\ 7s^3+18s\ (s.\tfrac{49}{10}$$
$$7s^3+\frac{147}{5}s$$
$$-\frac{57}{5}s)\ \frac{10s^2}{7}+6\ (-\tfrac{50}{7\times 57}s$$
$$\frac{10s^2}{7}$$
$$6)-\frac{57}{5}s\ (-\tfrac{57}{30}s$$
$$-\frac{57}{5}s$$
$$\times$$

Since the continued fraction contains the negative quotient, hence the **given polynomial is not Hurwitz.**

(*b*) $P(s)=s^4+s^3+6s^2+3s+6=M(s)+N(s)$

where $M(s)=s^4+6s^2+6;\quad N(s)=s^3+3s.$

Let us now perform the continued fraction expansion of $P(s)$ by dividing $M(s)$ by $N(s)$ and then inverting and dividing again as given below :

$$s^3+3s)\ s^4+6s^2+6\ (s$$
$$s^4+3s^2$$
$$3s^2+6)\ s^3+3s\ (\tfrac{1}{3}s$$
$$s^3+2s$$
$$s)\ 3s^2+6\ (3s$$
$$3s^2$$
$$6)\ s\ (\tfrac{s}{6}$$
$$s$$
$$\times$$

The continued fraction is then

$$P(s)=s+\cfrac{1}{\cfrac{s}{3}+\cfrac{1}{3s+\cfrac{1}{\frac{s}{6}}}}.$$

Since all the quotient terms of the continued fraction expansion are positive, **P(s) is Hurwitz**.

(*c*) $P(s)=s^4+s^3+2s^2+4s+1=M(s)+N(s)$

where $M(s)=s^4+2s^2+1;\quad N(s)=s^3+4s$

Using the method of continued fraction, The continued fraction yields

$$s^3+4s)\ s^4+2s^2+1\ (s$$
$$s^4+4s^2$$
$$-2s^2+1)\ s^3+4s\ (-\tfrac{s}{2}$$
$$s^3-\frac{s}{2}$$
$$\frac{9s}{2})\ -2s^2+1\ (-\tfrac{4}{9}s$$
$$-2s^2$$
$$1)\ \frac{9s}{2}\ (\tfrac{9s}{2}$$
$$\frac{9s}{2}$$
$$\times$$

The given **polynomial is not Hurwitz** because of presence of the negative quotient in the continued fraction.

EXAMPLE 21.7 *Check whether a polynomial expressed as $P(s)=s^3+6s^2+11s+6$ is Hurwitz or not.*

SOLUTION. Separating $P(s)$ into odd and even parts,

$$M(s)=s^3+11s;\quad N(s)=6s^2+6$$

The continued fraction expansion of $M(s)/N(s)$ is obtained as follows :

$$6s^2+6)\ s^3+11s\ (\tfrac{1}{6}s$$
$$s^3+s$$
$$10s)\ 6s^2+6\ (\tfrac{3}{5}s$$
$$6s^2$$
$$6)\ 10s\ (\tfrac{5}{3}s$$
$$10s$$
$$\times$$

Since the quotients in the continued fraction method become all positive, **the polynomial is Hurwitz.**

EXAMPLE 21.8 *Test whether the polynomial s^5+s^3+s is Hurwitz or not.*

SOLUTION. $P(s)(=s^5+s^3+s)$ consists of only odd functions, as a result of which it is not possible to perform the continued fraction expansion. Let us then take the derivative of $P(s)$. Assume derivative of $P(s)$ as $P'(s)$],

i.e., $$P(s)=5s^4+3s^2+1$$

Let $$\psi(s)=\frac{P(s)}{P'(s)}=\frac{s^5+s^3+s}{5s^4+3s^2+1}$$

and let us now proceed with continued fraction method as shown below :

$$5s^4+3s^2+1)\ s^5+s^3+s\ (\tfrac{s}{5}$$
$$s^5+\frac{3}{5}s^3+\frac{s}{5}$$
$$\frac{2}{5}s^3+\frac{4}{5}s)\ 5s^4+3s^2+1\ (\tfrac{25}{2}s$$
$$5s^4+10s^2$$
$$-7s^2+1)\ \frac{2}{5}s^3+\frac{4}{5}s\ (-\tfrac{2}{35}s$$
$$\frac{2}{5}s^3-\frac{2}{35}s$$
$$\frac{6}{7}s)\ -7s^2+1\ (-\tfrac{49}{6}s$$
$$-7s^2$$
$$1)\ \frac{6s}{7}\ (\tfrac{6s}{7}$$
$$\frac{6s}{7}$$
$$\times$$

In the continued fraction of the above function, the third and fourth quotients are negative. Hence, the **given polynomial is not Hurwitz.**

EXAMPLE 21.9 *Test whether the following functions are Hurwitz or not :*

(*i*) $s^5+3s^4+3s^3+4s^2+s+1$

(*ii*) s^4+3s^2+2

SOLUTION. (*i*)

$$P(s)=s^5+3s^4+3s^3+4s^2+s+1=M(s)+N(s)$$

where $M(s)=3s^4+4s^2+1;$

$N(s)=s^5+3s^3+s$

[$\because$ degree of $N(s)$ is higher than that of $M(s)$]

Let $$Z(s)=\frac{N(s)}{M(s)}$$

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EXAMPLE 21.15 *Find the range of values of m in P(s), so that P(s) is Hurwitz.*

Assume $P(s) = 2s^4 + s^3 + ms^2 + s + 2$.

SOLUTION.

$$P(s) = 2s^4 + s^3 + ms^2 + s + 2$$
$$= (2s^4 + ms^2 + 2) + (s^3 + s) = M(s) + N(s)$$

Applying the continued fraction expansion,

$$s^3 + s\ )\ 2s^4 + ms^2 + 2\ (\ 2s$$
$$2s^4 + 2s^2$$
$$(m-2)s^2 + 2\ )\ s^3 + s\ \left(\frac{s}{m-2}\right.$$
$$s^3 + \frac{2s}{m-2}$$
$$\left(1 - \frac{2}{m-2}\right)s$$

The quotients of the continued fraction expansion would be +ve only when $m \geq 2$ and $1 > \frac{2}{m-2}$ *i.e.*, $m > 4$.

Thus the range is $m > 4$ for **P(s) to be Hurwitz.**

21.5 POSITIVE REAL (PR) FUNCTIONS

It has already been known to us that the driving point impedance function $[Z(s)]$ as well as the driving point admittance function $[Y(s)]$ of one port network can be represented in the form of

$$F(s) = \frac{A(s)}{B(s)}$$
$$= \frac{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}$$

The function F(s) is called a *positive real (or PR) function** iff

(*a*) F(s) is *real* for s real

(*b*) B(s) is *Hurwitz* polynomial

(*c*) If F(s) has *poles* on ($j\omega$) axis, the poles are *simple* and the residues thereof are *real* and *positive*.

(*d*) Real $F(j\omega) \geq 0$ for all values of ω

[*i.e.*, when real part of is greater then zero, $\text{Re}(F(s) = \sigma > 0]$

To illustrate the concept of PR function in practical circuit theory, let us analyse the following examples of PR functions :

(*a*) Let $F(s)$ [$= Ls$, L being real and positive] be a PR function. If F(s) denotes an *impedance function, L becomes an inductance.*

(*b*) Let $F(s)$ [$= R$, R being real and positive] be another PR function. F(s) being denoting an *impedance*, R is a resistance.

(*c*) $F(s) = \left[\frac{K}{s}, K \text{ being real and positive}\right]$ be a PR function, when s is real, F(s) is real. Also when $\text{Re}(s) > 0$, $\text{Re}\,F(s) > 0\ [= \sigma]$

$$i.e., \ \text{Re}\left(\frac{K}{s}\right) = \frac{K.\sigma}{\sigma^2 + \omega^2} > 0$$

If $F(s)$ represents an impedance, the corresponding element becomes a *capacitor of 1/K Farads.*

We thus find that the passive impedances are all PR *functions. Similarly, the admittance functions are also* PR *functions. Also, all driving point immittances of passive networks are* PR *functions.*

21.5.1 Properties of PR Functions

Let $F(s) = \frac{A(s)}{B(s)}$

The properties of PR functions are as below :

(*i*) Both $A(s)$ and $B(s)$ polynomials are *Hurwitz.* They may have factors of the form $(s^2 + \omega^2)$ *i.e.*, the poles and zeros of a PR function cannot have +ve real parts *i.e.*, they cannot be in the right half of the s-plane. Only simple poles with +ve real residues can exist on the $j\omega$ axis.

(*ii*) The highest and lowest powers of $A(s)$ and $B(s)$ differ by *unity*. This condition prohibits multiple pole and zeros at $s = \infty$ and at $s = 0$ respectively.

(*iii*) If $F(s)$ is a PR function, the reciprocal of $F(s)$ is also a *PR function* (this implies that if the driving point impedance is a *PR function*, the driving point admittance is also a *PR function*).

(*iv*) The *sum* of PR functions is also a PR function (though the *difference* of two PR functions is not necessarily a PR function)

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* A real function whose real part is positive for values of s with positive real part is called a *positive real function*

SOLUTION. Let us apply the tests of PR function :

(i) Since all the coefficients of the polynomials in the numerator and denominator are +ve hence $F(s)$ is real if s is real.

(ii) It is also evident that the pole of the function lies on the left half of the s-plane [the zeros, $(-5 \pm \sqrt{21})$ also lie on the left half of the s plane].

(iii) $\text{Re}[F(j\omega)]$

$$= \text{Re}\left[\frac{-\omega^2 + 10 j\omega + 4}{j\omega + 2}\right]\left[\frac{-j\omega + 2}{-j\omega + 2}\right]$$

$$= \text{Re}\left[\frac{-2\omega^2 + 20 j\omega + 8 + j\omega^3 + 10\omega^2 - 4 j\omega}{\omega^2 + 4}\right]$$

$$= \text{Re}\left[\frac{8\omega^2 + 16 j\omega + j\omega^3 + 8}{\omega^2 + 4}\right]$$

$$= \text{Re}\left[\frac{(8\omega^2 + 8) + j(\omega^3 + 16\omega)}{\omega^2 + 4}\right] = \frac{8\omega^2 + 8}{\omega^2 + 4}$$

Since for all values of ω, $Re[Z(j\omega)] \geq 0$.

The function $Z(s)$ is **thus a PR function.**

EXAMPLE 21.21 *Check the positive realness of the function*

$$Y(s) = \frac{s^2 + 2s + 20}{s + 10}.$$

SOLUTION. Let us apply the tests of PR function to the given function.

In the function $Y(s)$, all the quotient terms are real and for s to a real quantity $Y(s)$ is real. Also the poles and zeros are on the left half of the s-plane for the given function.

Next, let us see the positive realness of the given function in the $(j\omega)$ domain.

$$\text{Re}[Y(j\omega) = \text{Re}\left[\frac{(j\omega)^2 + 2(j\omega) + 20}{j\omega + 10}\right]$$

$$= \text{Re}\left[\frac{-\omega^2 + 2 j\omega + 20}{j\omega + 10}\right]\left[\frac{-j\omega + 10}{-j\omega + 10}\right]$$

$$= \text{Re}\left[\frac{+ j\omega^3 + 2\omega^2 - 20 j\omega - 10\omega^2 + 20 j\omega + 200}{\omega^2 + 100}\right]$$

$$= \left[\frac{-8\omega^2 + 200 + j\omega^3}{\omega^2 + 100}\right] = \frac{-8\omega^2 + 200}{\omega^2 + 100}$$

Since for all values of ω, $\text{Re}[Y(j\omega)] \not\geq 0$ *i.e.*, this test certifies that the **function is not a positive real function.**

EXAMPLE 21.22 *A function is given by*

$$Z(s) = \frac{s^3 + 5s^2 + 9s + 3}{s^3 + 4s^2 + 7s + 9}.$$

Find the positive realness of the function.

SOLUTION. Let us proceed with the testing of the function for positive realness—

(i) Since all the coefficients in the numerator and denominator are having +ve values hence, for real value of s, $Z(s)$ is real.

(ii) To find whether the poles are on the left half of the s-plane, let us apply the Hurwitz criterion to the denominator using continued fraction method.

Let $P(s) = s^3 + 4s^2 + 7s + 9 = M_2(s) + N_2(s)$

where $M_2(s) = 4s^2 + 9$ and $N_2(s) = s^3 + 7s$.

Application of continued fraction method is shown below.

$$\psi(s) = \frac{N_2(s)}{M_2(s)} = \frac{s^3 + 7s}{4s^2 + 9}$$

$$4s^2 + 9\ )\ s^3 + 7s\ \left(\frac{s}{4}\right.$$
$$s^3 + \frac{9s}{4}$$
$$\frac{19s}{4}\ )\ 4s^2 + 9\ \left(\frac{16}{19}s\right.$$
$$4s^2$$
$$9\ )\ \frac{19s}{4}\ \left(\frac{19s}{4 \times 9}\right.$$
$$\frac{19s}{4}$$
$$\times$$

Since all the quotients are +ve in the continued fraction expansion, hence, the polynomial of $Z(s)$ in the denominator is Hurwitz.

(iii) In order to find whether $\text{Re}\, Z(j\omega) \geq 0$ for all ω, let us adopt slightly more mathematical manipulation.

Let $$Z(s) = \frac{M_1(s) + N_1(s)}{M_2(s) + N_2(s)}$$

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21.7 SUMMARY OF PROCEDURE OF SYNTHESIS

Step 1

The given function being a PR function, it may be either in form of $Z(s)$ or $Y(s)$. In this first step substitute $j\omega$ for s in the expression of $Z(s)$ or $Y(s)$, rationalise and find the real part of the function. Find the minimum value of the real part $[ReF(j\omega)]_{min}$. [This is done by differentiating the real part of the rationalised function with respect to ω and equating it to zero and obtaining the value of ω for which the real part of the function would be minimum. Substituting the value of ω in the real part of the rationalised function find $Re[F(j\omega)]_{min}$.].

Step 2

If the functions given in form of $Z(s)$ the $[Re\,Z(j\omega)]_{min}$ is then a constant term signifying the presence of a series resistance whose magnitude is equal to the minimum real value just obtained. However if given in form of $Y(s)$, the obtained min. value is that of a shunt resistor.

Step 3

In case the given function does not have any real part, then it becomes evident that the given function [$Z(s)$ or $Y(s)$] does not have any resistive part and contains either inductance or capacitance.

Step 4

In this step check whether the numerator of the given function has one 's' term higher. If the numerator has one 's' term higher, pole appears at $s=\infty$; on the other hand if the denominator has one s term higher, pole appears at $s=0$ This can be explained as follows :

Say $Z(s) = \dfrac{a_{n+1}s^{n+1} + a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$

By long division,

$$Z(s) = K_0 s + \frac{\text{Remainder}}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} \qquad \ldots(21.21)$$

It can now be observed that $Z(s)$ will have a pole at $s=\infty$.

On the other hand, if

$$Z(s) = \frac{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + a_n s^n}{b_1 s + b_2 s^2 + \ldots + b_n s^n},$$

by long division,

$$Z(s) = \frac{K_0}{s} + \frac{\text{remainder}}{b_1 + b_2 s + \ldots + b_{n-1}s^{n-1}} \qquad \ldots(21.22)$$

It can be observed that $Z(s)$ will have a pole at $s=0$.

In the former case, if the original function is expressed in impedance form, the first term then becomes a series inductor; if the original expression of the function is in admittance form, the former can reveal the presence of a shunt capacitor.

On the other hand, for the case when there is one higher s term in the denominator, the first term is a series capacitor provided the function is expressed in impedance form. If the function is expressed in admittance form, this case reveals the presence of a shunt inductor.

Step 5

Perform the first long division to obtain the first quotient. In the first case of step 4, the quotient will be (K_0 s) indicating the value of the series inductor (if the function is originally expressed in Z form) to be K_0 Henry. If the function is originally expressed in Y form, ($K_0 s$) indicates the presence of a shunt capacitor of value K_0 Farad. On the other hand if the first quotient is $\frac{K_0}{s}$ in step 4, the magnitude of the series capacitor is $(1/K_0)$ Farad when the original function is expressed in Z form. If the original function be expressed in Y form, this $\frac{K_0}{s}$ quotient indicates that the first element being shunt inductor, its value is $(1/K_0)$ Henry.

Step 6

Once the first part of the given function is realised, the remainder is inverted and the realisation is again commenced starting from step-1.

This process continuous till the last element is realised.

EXAMPLE 21.24 The driving point impedance of a network is given by

$$Z(s) = \frac{s^3 + 4s}{s^2 + 2}. \text{ Realise the network.}$$

SOLUTION. Step 1. Since the degree of numerator polynomial is one higher than that of denominator polynomial hence it is evident that $Z(s)$ will have a pole at $s=\infty$ indicating the presence of a series inductor whose value can be determined by long division of the numerator of $Z(s)$ by the denominator.

Step 2. $s^2+2)\,s^3+4s\,(s$

$\quad s^3+2s$

$\quad\quad 2s$

i.e., $Z(s) = s + \dfrac{2s}{s^2+2} = Z_1(s) + Z_2(s)$

Thus $Z_1(s)[=1.s]$ indicates that the series inductance would have value of 1 H.

Step 3. Since $Z_2(s) = \dfrac{2s}{s^2+2}$, $Y_2(s) = \dfrac{s^2+2}{2s}$

Presence of pole at $s=\infty$ is evident as the degree of numerator is still one higher. For the admittance function $Y_2(s)$, presence of pole at $s=\infty$ indicates a parallel capacitance whose value can be determined by breaking $Y_2(s)$ in partial fraction.

Step 4. $Y_2(s) = \dfrac{s}{2} + \dfrac{1}{s} = Y_3(s) + Y_4(s)$

$Y_3(s) = \left(\dfrac{1}{2}s\right)$ indicates the value of the capacitance to be $\dfrac{1}{2}$ Farad in parallel to $Y_4(s)\left(=\dfrac{1}{s}\right)$ which is evidently an inductor of L=1 Henry. The **complete realisation is shown in Fig. E21.1.**

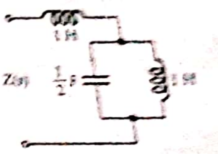

Fig. E21.1

CHAPTER 21

EXAMPLE 21.25 Realise the network whose impedance is given as $Z_1(s) = \dfrac{s^4+10s^2+7}{s^3+2s}$

SOLUTION. *Step 1.* Substitution of $s = j\omega$ reveals that $Z_1(s)$ does not have any real part and hence it can be inferred that $Z_1(s)$ would not have any resistance element.

Since the degree of s term in numerator is one higher than the degree of s in denominator hence it is evident that $Z_1(s)$ has a pole at infinity indicating presence of an series inductor. The magnitude of the series inductance can be obtained by using long division method.

Step 2. $s^3 + 2s\ )\ s^4 + 10s^2 + 7\ (\ s$

$\quad s^4 + 2s^2$

$\quad \overline{8s^2 + 7}$

$$\therefore Z_1(s) = \frac{s^4 + 10s^2 + 7}{s^3 + 2s} = s + \frac{8s^2 + 7}{s^3 + 2s}$$

$$= Z_2(s) + Z_3(s)$$

Thus the inductor has an impedance of $Z_2(s)$ being denoted by s *i.e.*, the inductance is 1 Henry.

Step 3. Since $Z_3(s) = \dfrac{8s^2 + 7}{s^3 + 2s}$, inverting it we get.

$$Y_3(s) = \frac{s^3 + 2s}{8s^2 + 7}.$$

Also for $Y_3(s)$, the degree of s in numerator is one more ;

$\therefore Y_3(s)$ would have a pole at infinity resulting a parallel capacitor.

Step 4. Again by long division, we find,

$8s^2 + 7\ )\ s^3 + 2s\ (\ \frac{s}{8}$

$\quad s^3 + \frac{7}{8}s$

$\quad \overline{\frac{9}{8}s}$

$$\therefore \quad Y_3(s) = Y_4(s) + Y_5(s)$$

$$= \frac{s}{8} + \frac{(9/8)s}{8s^2 + 7}.$$

i.e., $Y_4(s)$ is the admittance of the capacitor being given by $\frac{s}{8}$. This indicates than the capacitor has a capacitance of $\frac{1}{8}$ Farad.

Step 5. Next, converting $Y_5(s)$ to impedance form,

$$Z_5(s) = \frac{8s^2 + 7}{(9/8)s} = \frac{64}{9}s + \frac{56}{9s} = Z_6(s) + Z_7(s)$$

Since expression of $Z_5(s)$ has s in numerator one degree higher than that in denominator, hence it is evident that $Z_6(s)$ is a series inductor having inductance of $\frac{64}{9}$ H

Step 6. Again inversion of $Z_7(s)$ in admittance form $\left[Y_7(s) = \frac{9s}{56}\right]$ reveals the presence of a pole at infinity indicating the presence of a shunt capacitor of $\frac{9}{56}$ Farad.

The complete synthesis is shown in Fig. E21.2

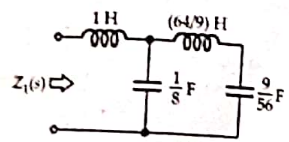

Fig. E21.2

EXAMPLE 21.26 *Realise the network having impedance function*

$$Z(s) = \frac{s^2 + 4s + 40}{s(s + 10)}.$$

SOLUTION. *Step 1.* Inspection of $Z(s)$ reveals that the pole is at origin ($s = 0$) for the given impedance function.

$\therefore$ Location of a pole at $s = 0$ indicates the presence of a series capacitor whose value can be determined using long division of numerator of $Z(s)$ by its denominator (Step 2).

Step 2. $s^2 + 10s\ )\ s^2 + 4s + 40\ (\ \frac{4}{s}$

$\quad 4s + 40$

$\quad s^2$

i.e.
$$Z(s) = \frac{s^2 + 4s + 40}{s(s + 10)} = \frac{4}{s} + \frac{s^2}{s^2 + 10s}$$

$$= \frac{4}{s} + \frac{s}{s + 10} = Z_1(s) + Z_2(s)$$

Then $Z_1(s) = \frac{4}{s}$ indicates the capacitor having $C_1 = \frac{1}{4}$ Farad in series with $Z_2(s)$ [Fig. E21.3(*a*)]

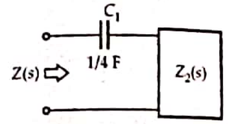

Fig. E21.3 (*a*)

Step 3. We will now realise $Z_2(s)$ by first inverting it as $Y_2(s)$.

$$\therefore \quad Y_2(s) = \frac{1}{Z_2(s)} = \frac{s + 10}{s}.$$

Since $Y_2(s)$ has a pole at $s = 0$, it is evident that this indicates a parallel inductance determined by dividing the numerator $(s + 10)$ by s (step 4).

Step 4. $s\ )\ s + 10\ (\ 1$

$\quad s$

$\quad 10$

i.e.,
$$Y_2(s) = \frac{s + 10}{s} = 1 + \frac{10}{s}$$

$$= \frac{10}{s} + 1 = Y_3(s) + Y_4(s)$$

$\therefore Y_3(s) = \frac{10}{s}$ indicates an inductor in parallel to $Y_4(s)$. Obviously if $Y_3(s) = \frac{10}{s}$, for an inductor, $Z_3(s) = \frac{s}{10} = L_3 s$ indicating $L_3 = \frac{1}{10}$ Henry.

The circuit is realised in Fig. E21.3(*b*) :

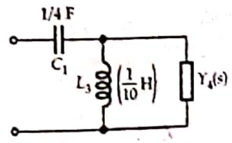

Fig. E21.3 (*b*)

Step 5. Removing $Y_3(s)$ from $Y_2(s)$ it is evident that the left out portion is $Y_4(s)[=1]$. Since it is independent of s term hence it is evident that it is a constant term representing a resistance of 1Ω.

Step 6. **The complete realisation is then shown below (Fig. E21.3(*c*))**

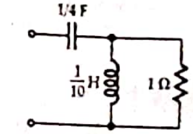

Fig. E21.3 (*c*)

EXAMPLE 21.27 *Realise the network having impedance function :*

$$Z(s) = \frac{s^2 + 2s + 10}{s(s + 5)}.$$

SOLUTION. *Step 1.* We observe that $Z(s)$ has a pole at $s = 0$. Location of a pole at $s = 0$ indicates the presence of a series capacitor whose value can be determined using long division of numerator of $Z(s)$ by its denominator (Step 2).

Step 2. $s^2 + 5s\ )\ s^2 + 2s + 10\ (\ \frac{2}{s}$

$\quad 2s + 10$

$\quad s^2$

$$\therefore \quad Z(s) = \frac{2}{s} + \frac{s^2}{s^2 + 5s} = \frac{2}{s} + \frac{s}{s + 5}$$

$$= Z_1(s) + Z_2(s)$$

Thus $Z_1(s) = \frac{2}{s}$ indicates the value of the series capacitance to be $\frac{1}{2}$ Farad.

Step 3. To realise $Z_2(s)$, it is inverted to gives $Y_2(s)$.

$$Y_2(s) = \frac{s + 5}{s} = 1 + \frac{5}{s}$$

$$= \frac{5}{s} + 1 = Y_3(s) + Y_4(s)$$

Since $Y_2(s)$ has a pole at $s = 0$, it is then evident that there exists a parallel inductance in $Y_3(s)$. Its value is indicated in step 4.

Step 4. $Z_3(s) = \dfrac{1}{Y_3(s)} = \dfrac{s}{5} = \dfrac{1}{5} \cdot s (= L_3 s)$

Thus the parallel inductance has a value of $\frac{1}{5}$ Henry.

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Step 5. The only left portion being $Y_4(s)$, in this problem it is 1Ω. Thus $Y_4(s) = 1$ indicates $Z_4(s) = 1$ i.e., presenting a resistor of 1Ω in parallel to $Y_3(s)$. **The complete realisation is shown in Fig. E21.4.**

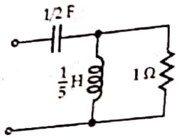

Fig. E21.4

EXAMPLE 21.28 *The impedance function of a network is given by*

$$Z(s) = \frac{6s^3 + 5s^2 + 6s + 4}{2s^3 + 2s}$$

Realise the network.

SOLUTION.

$$Z(s) = \frac{6s^3 + 5s^2 + 6s + 4}{2s^3 + 2s}$$

$$= \frac{6s^3 + 5s^2 + 6s + 4}{s(2s^2 + 2)}.$$

It may be observed that $Re[Z(s)]$ at $s = j\omega$ has terms devoid of s terms and its minimum real value is a constant term with $\omega = 0$. Thus the first element is a resistance term whose value can be found out by long division method.

$$2s^3 + 2s\;)\;6s^3 + 5s^2 + 6s + 4\;(\;3$$
$$\underline{6s^3 + 6s}$$
$$5s^2 + 4$$

$$\therefore\; Z(s) = 3 + \frac{5s^2 + 4}{2s^3 + 2s} = Z_1(s) + Z_2(s)$$

Thus, $Z_1(s)$ being realised as a s-free quantity, if is obviously a series resistor (of 3Ω).

Next we observe $Z_2(s)$. It is inverted and then

$$Y_2(s) = \frac{2s^3 + 2s}{5s^2 + 4}$$

$Y_2(s)$ has no real part and hence it would not have any resistor. Again, since the numerator has one degree higher than denominator, $Y_2(s)$ has a pole at $s = \infty$ indicating the presence of a parallel capacitor.

By long division,

$$5s^2 + 4\;)\;2s^3 + 2s\;(\;\frac{2s}{5}$$
$$\underline{2s^3 + \frac{8s}{5}}$$
$$\frac{2s}{5}$$

$$\therefore\; Y_2(s) = \frac{2s}{5} + \frac{2s/5}{5s^2 + 4} = Y_3(s) + Y_4(s)$$

$Y_3(s) = \left(\frac{2}{5}.s\right)$ indicates that the parallel capacitance is of $\frac{2}{5}$ Farad.

Next, we invert $Y_4(s)$ to obtain $Z_4(s)$

$$Z_4(s) = \frac{5s^2 + 4}{2s/5} = \frac{25s^2 + 20}{2s}$$

Since the degree of s is higher in numerator than denominator, $Z_4(s)$ would have a pole at $s = \infty$. This indicates a series inductor.

Also, $Z_4(s) = 12.5s + \frac{10}{s} = Z_5(s) + Z_6(s)$

Thus the inductor $Z_5(s)$ would be of 12.5 Henry while $Z_6(s)$ would be a series capacitor of $\frac{1}{10}$ Farad.

The complete realisation is shown in Fig. E21.5.

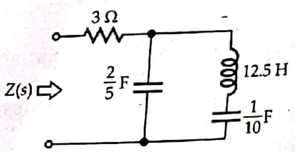

Fig. E21.5

EXAMPLE 21.29 *An admittance function is given as*

$$Y(s) = \frac{4s^2 + 6s}{s + 1}.$$

Realise the network.

SOLUTION. ***Step 1.*** We observe that the numerator polynomial is of one degree higher than the denominator polynomial. Thus a pole exists at $s = \infty$. This clearly indicates that for the given impedance function, for pole at $s = \infty$, a parallel capacitor exists whose value can be determined by long division method (Step-2)

Step 2.

$$s + 1\;)\;4s^2 + 6s\;(\;4s$$
$$\underline{4s^2 + 4s}$$
$$2s$$

i.e., $Y(s) = 4s + \frac{2s}{s + 1} = Y_1(s) + Y_2(s)$

$Y_1(s) = 4s$ indicates that the value of the capacitor would be 4 F.

Step 3. Next $Y_2(s)$ is inverted such that

$$Z_2(s) = \frac{s + 1}{2s} = \frac{1}{2} + \frac{1}{2s}$$

$$= \frac{1}{2s} + \frac{1}{2} = Z_3(s) + Z_4(s)$$

However, $Z_2(s)$ being having a pole at $s = 0$, it is evident that there will be a series capacitor represented by $Z_3(s)$ in series with $Z_4(s)$. Obviously $Z_3(s)$ being $\frac{1}{2s}$, the capacitor would be of 2 Farad.

Step 4. $Z_4(s)$ being devoid of s, it is a resistance of $\frac{1}{2}$Ω in series with $Z_3(s)$.

The complete realisation is shown in Fig. E21.6.

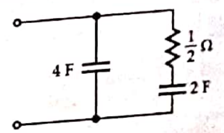

Fig. E21.6

EXAMPLE 21.30 *Synthesize* $Y(s) = \frac{7s + 5}{3s + 9}$.

SOLUTION. With $s = j\omega$, the real part of $Y(j\omega)$ appears which is obtained as

$$Re[Y(j\omega)] = \frac{21\omega^2 + 45}{9\omega^2 + 81}$$

The minimum real part occurs at $\omega = 0$.

$$\text{Min}[Re\,Y(j\omega)] = \frac{5}{9}.$$

Thus the resistive part in $Y(s)$ is obtained but since the expression is given in admittance form hence the value obtained is actually conductance instead of resistance. This, clearly indicates the resistance of $\frac{5}{9}$ ohm is connected in parallel as first element.

The left out portion of $Y(s)$ is thus

$$Y'(s) = Y(s) - \frac{5}{9} = \frac{7s + 5}{3s + 9} - \frac{5}{9}$$

$$= \frac{16s}{9s + 27}$$

$$\therefore\; Z(s) = \frac{9s + 27}{16s} = \frac{9}{16} + \frac{27}{16s}$$

i.e., the presence of a series resistor of $\frac{9}{16}$ ohm is obtained in this step.

The remaining portion is $\frac{27}{16s}$ which indicates a pole at zero. This reveals the identity of a series capacitor of $\frac{16}{27}$ Farad. **The complete realisation is shown in Fig. E21.7.**

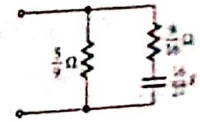

Fig. E21.7

21.8 REACTIVE NETWORKS

Any L-C one port network contains no resistive element and is thus dissipationless. The driving point immittance (either impedance or admittance) can be obtained by combining impedance or admittance expression for the elements of the network. LC networks are usually called the *reactive networks.*

Any given reactive network may be split up into either series or parallel combination of L-C elements. In series combination, the total impedance $Z(s)$ of the combination is given by

$$Z(s) = Z_1(s) + Z_2(s) + \ldots + Z_n(s) \quad \text{...(21.23)}$$

$$\text{or, } Z(s) = (L_1 + L_2 + \ldots + L_n)s + \left[\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}\right]\frac{1}{s} \quad \text{...(21.23(a))}$$

[∵ the inductor L has impedance Ls while a capacitor C has impedance $\frac{1}{Cs}$]