1)

Bessel's function (Recurrence formulae) → important for for main exam, (CE+ME) Module - 3

Formula 1, $\boxed{x J_n' = n J_n - x J_{n+1}}$

Proof:- we know that

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \lceil n+r+1} \left(\frac{x}{2}\right)^{n+2r}$$

Now differentiating w.r.t x we get

$$J_n' = \sum \frac{(-1)^r (n+2r)}{\lfloor r \, \lceil n+r+1} \left(\frac{x}{2}\right)^{n+2r-1} \times \frac{1}{2}$$

$$\text{Now } x J_n' = n \sum \frac{(-1)^r}{\lfloor r \, \lceil n+r+1} \left(\frac{x}{2}\right)^{n+2r} \times \cancel{1} + x \sum \frac{(-1)^r 2r}{2 \lfloor r \, \lceil n+r+1} \times \left(\frac{x}{2}\right)^{n+2r-1}$$

$$\Rightarrow x J_n' = n J_n + x \sum_{r=1}^{\infty} \frac{(-1)^r}{\lfloor r-1 \, \lceil n+r+1} \left(\frac{x}{2}\right)^{n+2r-1}$$ put $r-1 = s$

$$n J_n + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{\lfloor s \, \lceil n+s+2} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= n J_n - x \sum_{s=0}^{\infty} \frac{(-1)^s}{\lfloor s \, \lceil (n+1)+(s+1)} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$\boxed{x J_n' = n J_n - x J_{n+1}}$$ → Proved

(2) 2nd formula $x J_n' = -n J_n + x J_{n-1}$

$\Rightarrow x \sum_{r=0}^{\infty} (-1)$ ⇒

$\left(\frac{1}{2}\right)^{-1+1-1}$

$$\text{Now } x J_n' = x \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \lceil n+r+1} (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \times \frac{1}{2}$$

$$= n \sum_{r=1}^{\infty} \frac{(-1)^r}{\lfloor r \, \lceil n+r+1} \left(\frac{x}{2}\right)^{n+2r} \times \left(\frac{1}{2}\right)^{-1} \times \frac{1}{2}$$

(2)

$\boxed{\text{2nd formula}}$

$$x J_n' = -n J_n + x J_{n-1}$$

we have.

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_n' = \sum \frac{(-1)^r}{\lfloor r} \frac{(n+2r)}{\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \times \frac{1}{2}$$

$$\Rightarrow x J_n' = \cancel{x} \sum \frac{(-1)^r}{\lfloor r} \frac{(n+2r)}{\Gamma(n+r+1)} \frac{(x)^{n+2r}}{(2)^{n+2r-1}} \times \frac{1}{2}$$

$$= \sum \frac{(-1)^r}{\lfloor r} \frac{(n+2r)}{\Gamma(n+r+1)} \frac{(x)^{n+2r}}{(2)^{n+2r-1} \times 2^1} \times 2 \times \frac{1}{2}$$

$$= \sum \frac{(-1)^r}{\lfloor r} \frac{(n+2r)}{\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}.$$

$$\cancel{= n \sum \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum \frac{(-1)^r 2r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}}$$

$$= \sum \frac{(-1)^r}{\lfloor r} \frac{(2n+2r-n)}{\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum \frac{(-1)^r (2n+2r)}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - n \sum \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum \frac{(-1)^r \times 2 \cancel{(n+r)}}{\lfloor r \, \cancel{(n+r)} \Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r} - n J_n$$

$$\Rightarrow x \sum \frac{(-1)^r}{\lfloor r \, \Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} - n J_n$$

$$\Rightarrow x \sum \frac{(-1)^r}{\Gamma((n-1)+(r+1))} \left(\frac{x}{2}\right)^{(n-1)+2r} - n J_n$$

$$\Rightarrow \boxed{x J_{(n-1)} - n J_n} \rightarrow \text{Proved}$$

③

Formula-3 $\boxed{2J_n' = J_{n-1} - J_{n+1}}$

we know that

by above two formula

$$xJ_n' = nJ_n - xJ_{n+1}$$

and $$xJ_n' = -nJ_n + xJ_{n-1}$$

adding above

$$2xJ_n' = xJ_{n-1} - xJ_{n+1}$$

$$\Rightarrow \boxed{2J_n' = J_{n-1} - J_{n+1}} \rightarrow \text{Proved}$$

Formula

④ Show that $2nJ_n = x(J_{n-1} + J_{n+1})$

by above two formula

$$xJ_n' = nJ_n - xJ_{n+1} \quad \text{—(i)}$$

$$xJ_n' = -nJ_n + xJ_{n-1} \quad \text{—(ii)}$$

by (i) − (ii)

$$xJ_n' - xJ_n' = nJ_n - xJ_{n+1} + nJ_n - xJ_{n-1}$$

$$\Rightarrow 0 = 2nJ_n - xJ_{n+1} - xJ_{n-1}$$

$$\Rightarrow \boxed{2nJ_n = x(J_{n+1} + J_{n-1})}$$

Formula 5

$$\frac{d}{dx}\left(x^{-n}J_n\right) = -x^{-n}J_{n+1}$$

we know that $xJ_n' = nJ_n - xJ_{n+1}$

multiplying both side by

x^{-n-1} we obtain

$$x^{-n}J_n' = x^{-(n+1)}nJ_n - x^{-n-1+1}J_{n+1}$$

$$\Rightarrow x^{-n}J_n' = nx^{-n-1}J_n - x^{-n}J_{n+1}$$

$$\Rightarrow x^{-n}J_n' - nx^{-n-1}J_n = -x^{-n}J_{n+1}$$

$$\Rightarrow \boxed{\frac{d}{dx}\left(x^{-n}J_n\right) = -x^{-n}J_{n+1}} \rightarrow \text{Proved}$$

Date
Page

(7) Formula VI $\frac{d}{dx}\left(x^n J_n\right) = x^n J_{n-1}$

Proof:- we know that $x J_n' = -nJ_n + x J_{n-1}$

multiplying both side by x^{n-1}

$\Rightarrow x^n J_n' = -x^{n-1} n J_n + x^n J_{n-1}$

$\Rightarrow x^n J_n' + x^{n-1} n J_n = x^n J_{n-1}$

$$\Rightarrow \boxed{\frac{d}{dx}\left(x^n J_n\right) = x^n J_{n-1}}$$

Example on above ~~Reccar~~ Recurrence formula

Ex→1 Express J_5 in terms of J_1 and J_2

we know that by Reccurence formula

$2n J_n = x(J_{n-1} + J_{n+1})$

$\Rightarrow 2n J_n - x J_{n-1} = x J_{n+1}$ — (A) Put $n = 4$

$2 \times 4 J_4 - x J_3 = x J_5$

$\frac{8}{x} J_4 - \cancel{x} J_3 = J_5$ — (1) ~~in A Put~~

in (A) Put $n = 3$

$\Rightarrow x J_4 = 2 \times 3 J_3 - x J_2$

$J_4 = \frac{6}{x} J_3 - \cancel{x} J_2$

Put in (1)

$\frac{8}{x}\left(\frac{6}{x} J_3 - \cancel{x} J_2\right) - \cancel{x} J_3 = J_5$

$\left(\frac{48}{x^2} - 1\right) J_3 - \frac{8}{x} J_2 = J_5$ | Put in (A) $n = 2$

thus

$J_3 = \frac{2 \times 2}{x} J_2 - J_1$

$\Rightarrow \left(\frac{48}{x^2} - 1\right)\left(\frac{4}{x} J_2 - J_1\right) - \frac{8}{x} J_2 = J_5$

(5)

$$\Rightarrow \frac{192}{x^3} J_2 - \frac{48}{x^2} J_1 - \frac{4}{x} J_2 + J_1 - \frac{8}{x} J_2 = J_5$$

$$\Rightarrow \left(\frac{192}{x^3} - \frac{12}{x}\right) J_2 + \left(1 - \frac{48}{x^2}\right) J_1 = J_5$$

EX→2 Prove that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3-x^2}{x^2} \sin x - \frac{3\cos x}{x}\right)\right]$$

Proof:– from reccurence relation (4)

$$2n J_n = x J_{n-1} + x J_{n+1} \quad \text{Put}$$

$$J_{n+1} = \frac{2n}{x} J_n - J_{n-1} \quad \text{Put } n = 1/2$$

$$J_{3/2} = \frac{1}{x} J_{1/2} - J_{-1/2}$$

Again put $n = 3/2$

$$J_{5/2} = \frac{3}{x} J_{3/2} - J_{1/2}$$

$$\Rightarrow J_{5/2} = \frac{3}{x}\left[\frac{1}{x} J_{1/2} - J_{-1/2}\right] - J_{1/2}$$

$$\Rightarrow J_{5/2} = \frac{3}{x^2} J_{1/2} - \frac{3}{x} J_{-1/2} - J_{1/2}$$

$$\Rightarrow J_{5/2} = \left(\frac{3}{x^2} - 1\right) J_{1/2} - \frac{3}{x} J_{-1/2}$$

$$= \left(\frac{3}{x^2} - 1\right) \sqrt{\frac{2}{\pi x}} \sin x - \frac{3}{x} \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3-x^2}{x^2} \sin x - \frac{3\cos x}{x}\right)\right]$$

(i) Ex→3 Prove that following relation

$$x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x).$$

Soln:- We know that

$J_n(x)$ is the solution of (1)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

thus $x^2 J_n'' + x J_n' + (x^2-n^2)J_n = 0$ —(1)

we know that recurrence formula.

$$x J_n' = n J_n - x J_{n+1} \qquad \text{Put in (1)}$$

$$x^2 J_n'' + \cancel{x J_n'}\,(n J_n - x J_{n+1}) + (x^2-n^2)J_n = 0$$

$$\Rightarrow \boxed{x^2 J_n'' = (n^2 - n - x^2) J_n + x J_{n+1}} \quad \text{Proved}$$

Assignment

(I) Prove that $J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \lfloor 1 \lfloor 2} + \frac{x^5}{2^5 \lfloor 2 \lfloor 3} - \frac{x^7}{2^7 \lfloor 3 \lfloor 4} + \cdots$

(II) $\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x)$

Equations Reducible to "BESSEL'S EQUATION":-

(a) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = 0$ —(1)

Put $\boxed{t = kx}$ then solution of (1) is

$$y = C_1 J_n(t) + C_2 J_{-n}(t)$$

$$\boxed{\text{i.e. } y = C_1 J_n(kx) + C_2 J_{-n}(kx)}$$

(b) $x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + k^2xy = 0$ —(1)

Put $\boxed{y = x^n z;}$ $\frac{dy}{dx} = x^n\frac{dz}{dx} + n x^{n-1} z$

Solution of (1) is $y = x^n\left[C_1 J_n(kx) + C_2 J_{-n}(kx)\right]$, $n \neq 1$

where $n = \frac{1-a}{2}$

(c) equation of the type

$$x\frac{d^2y}{dx^2} + c\frac{dy}{dx} + k^2x^r y = 0$$

Put $\boxed{x = t^m,}$

it's solution is

$$y = x^{n/m}\left[C_1 J_n\left(km\, x^{1/m}\right) + C_2 J_{-n}\left(km\, x^{1/m}\right)\right]$$

ORTHOGIONALITY OF BESSEL FUNCTION

$$\int_0^1 x J_n(\alpha x)\cdot J_n(\beta x)\,dx = 0$$ where α and β are the roots of ~~Bessel function~~ $J_n(x) = 0$

Proof! - Let us consider two special type of differential equation, that is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\alpha^2x^2 - n^2)y = 0 \quad \text{--- (I)}$$

$$\Rightarrow x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (\beta^2x^2 - n^2)z = 0 \quad \text{--- (II)}$$

Solution of (I) and (II) are $y = J_n(\alpha x)$ and $y = J_n(\beta x)$ respectively.

Multiplying (I) by $\frac{z}{x}$ and (II) by $-\frac{y}{x}$ and adding, we get

$$\frac{z}{x}\left\{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\alpha^2x^2 - n^2)y\right\}$$

$$-\frac{y}{x}\left\{x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (\beta^2x^2 - n^2)z\right\} = 0$$

$$\Rightarrow x\left\{z\frac{d^2y}{dx^2} - y\frac{d^2z}{dx^2}\right\} + \left\{z\frac{dy}{dx} - y\frac{dz}{dx}\right\}$$

$$+\left(\frac{\alpha^2x^2yz}{x} - \frac{\beta^2x^2yz}{x}\right) + \left(\frac{-n^2yz}{x} + \frac{n^2yz}{x}\right) = 0$$

$$x\left\{z\frac{d^2y}{dx^2} - y\frac{d^2z}{dx^2}\right\} + \left\{z\frac{dy}{dx} - y\frac{dz}{dx}\right\} + (\alpha^2 - \beta^2)xyz = 0 \quad \text{--- (3)}$$

8

classmate
Date
Page

$\Rightarrow$ integrating w.r.t x ~~from~~ between limit 0 and 1, we get

$$\left[x\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)\right]_0^1+(\alpha^2-\beta^2)\int_0^1 xyz\,dx=0$$

$$\Rightarrow(\beta^2-\alpha^2)\int_0^1 xyz\,dx=\left[x\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)\right]_0^1$$

$$\Rightarrow(\beta^2-\alpha^2)\int_0^1 xyz\,dx=\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)_{x=1} \quad -(4)$$

since $y=J_n(\alpha x)\Rightarrow\frac{dy}{dx}=\alpha J_n'(\alpha x)$

and $z=J_n(\beta x)\Rightarrow\frac{dz}{dx}=\beta J_n'(\beta x)$

we get

$$\Rightarrow(\beta^2-\alpha^2)\int_0^1 x\{J_n(\alpha x)J_n(\beta x)\}dx=J_n(\beta x)\alpha J_n'(\alpha x)-J_n(\alpha x)\beta J_n'(\beta x) \quad -(5)$$

since α and β are roots of $J_n(x)=0$

thus $J_n(\alpha)=0$ and $J_n(\beta)=0$ put in (5)

$$\Rightarrow(\beta^2-\alpha^2)\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx=0$$

$$\Rightarrow\boxed{\int_0^1 x\,J_n(\alpha x)\times J_n(\beta x)\,dx=0}$$