(1)

BESSEL'S FUNCTIONS :- (CE+ME) (Math II) (2nd Sem) Module - 2

To understand this topic we should have to understand the concepts of GAMA functions.

so that at first we shall discuss some formula and concepts of GAMA function

(1) $\int_0^\infty e^{-x} x^{n-1}\,dx = \Gamma n$ read as gamma function of n

(a) $\Gamma 1 = 1$ (b) $\Gamma n+1 = n\Gamma n$ (c) $\Gamma n+1 = \lfloor n$ factorial n
(d) $\Gamma \frac{1}{2} = \sqrt{\pi}$

Evaluate $\Gamma -\frac{1}{2}$

Now $\Gamma -\frac{1}{2}+1 = -\frac{1}{2}\Gamma -\frac{1}{2}$ ⇒

$\Rightarrow \Gamma \frac{1}{2} = -\frac{1}{2}\Gamma \frac{-1}{2}$

$\Rightarrow \sqrt{\pi} = -\frac{1}{2}\Gamma \frac{-1}{2}$

$\Rightarrow -2\sqrt{\pi} = \Gamma -1/2$

$\Gamma \frac{9}{2} = \frac{7}{2}\Gamma \frac{7}{2} = \frac{7}{2}\frac{5}{2}\Gamma \frac{5}{2}$

$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma \frac{1}{2} = \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}$

BESSEL'S EQUATION

Particular type of differential equation in mathematics is $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ — (1)

which is known as Bessel's equation of order n. Its particular solutions are called Bessels functions.

Let $y = \sum_{r=0}^{\infty} a_r x^{m+r}$ ~~or~~ are solution of (1) → (2)

Now $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}$ and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$

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Substituting in equation (1) and find co-efficient of x^m which gives indicial equation

$$a_0[(m+0)^2 - n^2] = 0 \quad \text{where } a_0 \neq 0$$

thus $m^2 = n^2$

$\Rightarrow m = \pm n$

equating the co-efficient of next lowest degree ie x^{m+1} $\rightarrow$ we get $a_1 = 0$ since $(m+1)^2 - n^2 \neq 0$

highest co-efficient of x^{m+r+2} is zero we get Recurring formula

i.e $$a_{r+2} = -\frac{a_r}{(m+r+2)^2 - n^2}$$

$$\Rightarrow a_1 = a_3 = a_5 = 0 \ldots$$

but

$$a_2 = \frac{-a_0}{(m+2)^2 - n^2}$$

$$a_4 = -\frac{a_2}{(m+4)^2 - n^2} = \frac{+a_0}{\{(m+2)^2 - n^2\}\{(m+4)^2 - n^2\}}$$

$$a_6 = \frac{-a_0}{\{(m+2)^2 - n^2\}\{(m+4)^2 - n^2\}\{(m+6)^2 - n^2\}}$$

Put in eq^n (2)

thus $$y = a_0\left[x^m - \frac{1}{(m+2)^2 - n^2}x^{m+2} + \frac{1}{\{(m+2)^2 - n^2\}\{(m+4)^2 - n^2\}}x^{m+4} - \frac{1}{\{(m+2)^2 - n^2\}\{(m+4)^2 - n^2\}\{(m+6)^2 - n^2\}}x^{m+6} + \ldots\right]$$

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Thus $y = \sum_{r=0}^{\infty} a_0 x^m \left[1 - \frac{x^2}{(m+2)^2 - n^2} + \frac{1}{\{(m+2)^2 - n^2\}\{(m+4)^2 - n^2\}} x^4 - \cdots \right]$

for $m = n$

$$a_0 x^n \left[1 - \frac{x^2}{2 \times 2(n+1)} + \frac{x^4}{2 \times 2(n+1) \times 4 \times 2(n+2)} \right]$$

for $m = n$

$$a_0 x^n \left[1 - \frac{x^2}{\{(n+2)^2 - n^2\}} + \frac{x^4}{\{(n+4)^2 - n^2\} \times \{(n+2)^2 - n^2\}} - \frac{x^6}{\{(n+6)^2 - n^2\} \times \{(n+4)^2 - n^2\} \{(n+2)^2 - n^2\}} + \cdots \right]$$

$$= a_0 x^n \left[1 - \frac{x^2}{(n+2+n)(n+2-n)} + \frac{x^4}{(n+2+n)(n+2-n)(n+4+n)(n+4-n)} - \frac{x^6}{(n+2+n)(n+2-n)(n+4+n)(n+4-n)(n+6+n)(n+6-n)} + \cdots \right]$$

$$= a_0 x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2 (n+1) \times 2 (n+2)} - \frac{x^6}{4^3 \times 2 (n+1)(n+2) \times 3 (n+3)} + \frac{x^8}{4^4 \times 2 \times 3 \times 4 (n+1)(n+2)(n+3)(n+4)} - \cdots \right]$$

$$= a_0 x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2 \times \lfloor 2 (n+1)(n+2)} - \frac{x^6}{4^3 \times \lfloor 3 (n+1)(n+2)(n+3)} + \frac{x^8}{4^4 \times \lfloor 4 (n+1)(n+2)(n+3)(n+4)} - \cdots \right]$$

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Thus for $m=n$

$$y_1 = a_0 x^n \left[\quad a_0 x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r} \lfloor r (n+1)(n+2)(n+3) \cdots (n+r)} \right. \quad \text{— (A)}$$

for $m=-n$ — (B)

$$y_2 = a_0 x^{-n} \sum_{r=0} (-1)^r \frac{x^{2r}}{2^{2r} \lfloor r (-n+1)(-n+2)(-n+3) \cdots (-n+r)}$$

complete solution of (1) is

$$y = C_1 y_1 + C_2 y_2$$

case I [when n is not integer or zero

If we take $a_0 = \frac{1}{2^n \Gamma n+1}$ then y_1 is called the Bessel function of first kind order n → denoted by $J_n(x)$ and y_2 is called Bessel's function of ~~second~~ first kind order $-n$ → denoted by $J_{-n}(x)$

where clearly

NOW
$$J_n(x) = \frac{1}{2^n \Gamma n+1} \sum (-1)^r \frac{x^{n+2r}}{2^{2r} \lfloor r (n+1)(n+2) \cdots (n+r)}$$

$$= \frac{1}{2^n \Gamma n+1} \left[1 - \frac{x^{n+2}}{4(n+1)} + \frac{x^{n+4}}{4^2 \times \lfloor 2 (n+1)(n+2)} - \frac{x^{n+6}}{4^3 \times \lfloor 3 (n+1)(n+2)(n+3)} \cdots \right]$$

$$= \frac{1}{2^n \Gamma n+1} - \left(\frac{x}{2}\right)^n \times \frac{x^2}{4 \times (n+1) \Gamma n+1} + \left(\frac{x}{2}\right)^n \frac{x^4}{4^2 \times \lfloor 2 (n+2)(n+1) \Gamma n+1}$$

$$- \left(\frac{x}{2}\right)^n \frac{x^6}{4^3 \times \lfloor 3 (n+3)(n+2)(n+1) \Gamma n+1} + \cdots$$

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$$= \frac{1}{2^n \Gamma n+1} - \left(\frac{x}{2}\right)^{n+2} \times \frac{1}{\Gamma n+2} + \left(\frac{x}{2}\right)^{n+4} \times \frac{1}{\lfloor 2 \, \Gamma n+3} - \left(\frac{x}{2}\right)^{n+6} \frac{1}{\lfloor 3 \, \Gamma n+4} + \cdots$$

Thus $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma n+r+1} \left(\frac{x}{2}\right)^{n+2r}$

$||^{ly}$ $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma -n+r+1} \left(\frac{x}{2}\right)^{-n+2r}$

Thus complete solution of Bessel's equation (1) is

$$y = A J_n(x) + B J_{-n}(x)$$

Now

Case I when n is not integer or zero thus Bessel's function or solution of Bessel's equation is

$$\boxed{y = A J_n(x) + B J_{-n}(x)}$$

Case II if $n=0$ then $y_1 = y_2$ then complete solution of (1) Bessel's equation is Bessel's function of order zero

i.e $J_0(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma r+1} \left(\frac{x}{2}\right)^{2r}$

$$= \left[\frac{1}{\Gamma 1} - \left(\frac{x}{2}\right)^2 \frac{1}{\lfloor 2} + \left(\frac{x}{2}\right)^4 \frac{1}{\lfloor 3} - \cdots\right]$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} - \frac{x^6}{2^2 \times 4^2 \times 6^2}$$

(6) <u>When n is an integer</u> :→ y_2 fails for +ve value of n and y_1 fails for -ve value of n

Now we have to find independent solution of

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad \text{—(1)}$$

For this

<u>When n is integer</u>

$$y = u\,J_n(x) \quad \text{—(A)}$$ be solution of (1), we

obtain
$$y' = u'J_n + uJ_n'$$
$$y'' = u''J_n + J_n'u' + u'J_n' + uJ_n''$$
$$= u''J_n + 2u'J_n' + uJ_n''$$

Put in eqn (1)

$$x^2\left(u''J_n+2u'J_n'+uJ_n''\right)+x\left(u'J_n+uJ_n'\right)$$
$$+\left(x^2-n^2\right)uJ_n \cancel{(x)} = 0$$

$$\Rightarrow u\{x^2J_n''+xJ_n'+(x^2-n^2)J_n\}+x^2u''J_n+2x^2u'J_n'+xu'J_n=0 \quad \text{—(2)}$$

Since J_n is a solution of (1) thus put

$x^2J_n''+xJ_n'+(x^2-n^2)J_n=0$ in (2)

we get

$$x^2u''J_n+2x^2u'J_n'+xu'J_n=0$$

dividing both sides by $x^2u'J_n$ we get

$$\frac{u''}{u'}+2\frac{J_n'}{J_n}+\frac{1}{x}=0$$

i.e $\frac{d}{dx}(\log u')+2\frac{d}{dx}(\log J_n)+\frac{d}{dx}(\log x)=0$

$$\Rightarrow \cancel{\log u'}\ \frac{d}{dx}\left\{\log(u'J_n^2x)\right\}=0$$

integrating both sides

$$\Rightarrow \log(u'J_n^2x)=\log B$$
$$\Rightarrow u'J_n^2x = B$$
$$\Rightarrow u' = \frac{B}{J_n^2x}$$

$$u = B\int\frac{dx}{J_n^2x}+A$$

Put in (A)

$$y = \left\{ B\int \frac{dx}{J_n^2 x} + A \right\} J_n$$

$$\Rightarrow y = A J_n + B J_n \int \frac{dx}{x J_n^2}$$

$$\Rightarrow y = A J_n(x) + B J_n(x) \int \frac{dx}{x \{J_n(x)\}^2}$$

$$y = A J_n(x) + B Y_n(x)$$

where $Y_n(x) = J_n(x) \int \dfrac{dx}{x\{J_n(x)\}^2}$

Here $Y_n(x)$ is called Bessel's function of ~~order~~ the second kind of order n

or Neumann function

So that in conclusion

(i) when n is not integer or zero Bessel's function is

$$y = A J_n(x) + B J_{(-n)}(x)$$

(ii) when n = 0 then Bessel's function is

$$y = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} - \frac{x^6}{2^2 \times 4^2 \times 6^2} + \cdots$$

(iii) when n is an integer then Bessel's function is

$$y = A J_n + B J_n(x) \int \frac{dx}{x\{J_n(x)\}^2}$$

(8) Now in Neumann function i.e in Y_n

if n is an integer then

$$\rightarrow Y_n(x) = \lim_{m \to n} \left[\frac{1}{\sin m\pi} \left\{ J_m(x) \cos m x - J_{-(m)}(x) \right. \right]$$

when n is not an integer

$$\rightarrow Y_n(x) = \frac{1}{\sin n\pi} \left[J_n(x) \cos n\pi - J_{-n}(x) \right]$$