of lifting vanes, control surfaces and propellers. Indeed, the enunciation of boundary layer theory provided the key that unlocked the door for much of the progress during this century in the fields of both fluid mechanics and convective heat transfer.

The boundary layer investigations are usually concerned with the estimation of boundary layer thickness parameters, the shear stress and the associated drag on the solid surface.

10.1. DESCRIPTION OF BOUNDARY LAYER

Consider a continuous flow of fluid along the surface of a thin flat plate with its sharp leading edge set parallel to the flow direction.

The salient aspects of the flow situations are :

(*i*) The free stream flow has a uniform velocity U_0 in the *x*-direction. Particles of fluid adhere to the plate surface as they approach it and the fluid is slowed down considerably. The fluid becomes stagnant or virtually so in the immediate vicinity of the plate surface. Generally it is presumed that there is no slip between the fluid and the solid boundary. Thus, there exists a region where the flow velocity changes from that of solid boundary to that of the mainstream fluid, and in this region the velocity gradient exists in the fluid. Consequently the flow is rotational and shear stresses are present. This region of changing velocity has been called the **hydrodynamic boundary layer**, a concept first suggested by Ludwig Prandtl in the year 1904.

(*ii*) The condition $\frac{\partial u}{\partial y} \neq 0$ is a true for the zone within the boundary layer, whilst the conditions for flow beyond the boundary layer and at its outer border are :

$$\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad u = U_0$$

Thus all the variation in the velocity is concentrated in a comparatively thin layer in immediate vicinity of the plate surface.

(*iii*) The concepts of boundary layer thickness and outer edge of the boundary layer are quite ficticious as there is no abrupt transition from the boundary layer to the flow beyond or outside. Velocity within the boundary layer approaches the free stream velocity asymptotically. Usually the boundary layer thickness δ is taken to be the distance from the plate surface to a point at which the velocity is, say within 1 percent of the asymptotic limit, *i.e.*, $u = 0.99\ U_0$; δ then becomes a nominal measure of the thickness of the boundary layer, *i.e.*, of the region in which the major portion of the velocity deformation takes place. The thickness is measured normal to the plate surface. The boundary layer is normally very thin in comparison with the dimensions of the body immersed in the flow.

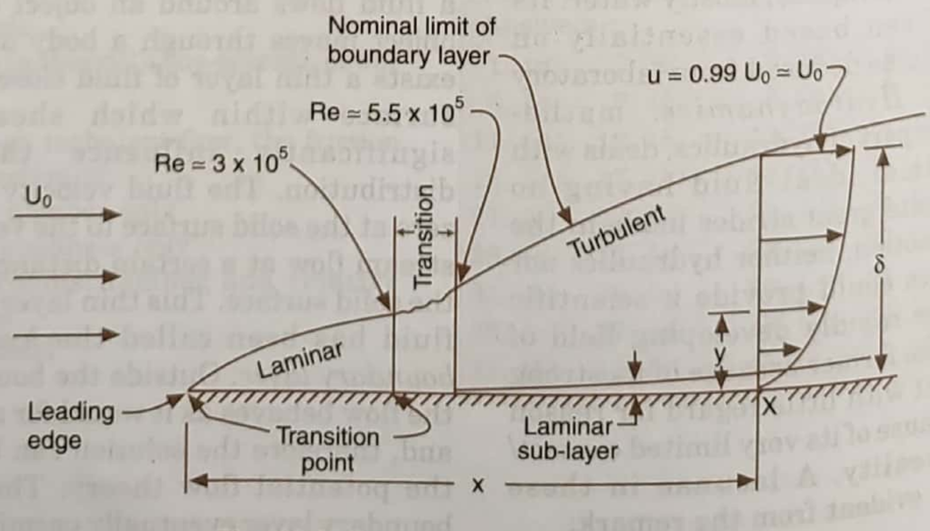

Fig. 10.1. Boundary layer on a flat plate

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(*iv*) The thickness of the boundary layer is variable along the flow direction ; it is zero at leading edge of the plate and increases as the distance x from the leading edge is increased. This aspect may be attributed to the viscous forces which dissipate more and more energy of the fluid stream as the flow proceeds. Consequently, a large group of the fluid particles is slowed down.

The boundary layer growth is also governed by other parameters such as the magnitude of the incoming velocity and the kinematic viscosity of the flowing fluid. For higher incoming velocities, there would be less time for viscous forces to act and accordingly there would be less quantum of boundary layer thickness at a particular distance from the leading edge. Further, the boundary layer thickness is greater for the fluids with greater kinematic viscosity.

(*v*) For some distance from the leading edge, the boundary layer is laminar and the velocity profile is parabolic in character. Flow within the laminar boundary layer is smooth and the streamlines are essentially parallel to the plate. Subsequently, the laminar boundary layer becomes unstable and the laminar flow undergoes a change in its flow structure at a certain point, called **transition point**, in the flow field. Within the transition zone, the flow is unstable and is referred to as transition flow. After going through a transition zone of finite length, the boundary layer entirely changes to turbulent boundary layer.

(*vi*) The turbulent boundary layer does not extend to the solid surface. Underlying it, an extremely thin layer called *laminar sublayer*, is formed wherein the flow is essentially of laminar character. Outside the boundary layer, the main fluid flow may be either laminar or turbulent.

(*vii*) The pattern of flow in the boundary layer is judged by the Reynolds number $R_e = \frac{x U_0}{\nu}$ where x is the distance along the plate and measured from its leading edge. The transition from laminar to turbulent pattern of flow occurs at values of Reynolds number between 3×10^5 to 5×10^5. Besides this critical Reynolds number, the co-ordinate points at which deterioration at the laminar layer begins and stabilized turbulent flow sets in is dependent on the surface roughness, plate curvature and the pressure gradient, and the intensity of turbulence of the free stream flow.

(*viii*) In laminar boundary layer, the velocity gradient becomes less steep as one proceeds along the flow. It is because now the change in velocity from no slip at the plate surface to free stream value in the potential core occurs over a greater transverse distance. Nevertheless in a turbulent boundary layer, there occurs an interchange of momentum and energy amongst the individual layers comprising the boundary layer. Consequently, a turbulent boundary layer has a fuller velocity profile and a much steeper velocity gradient at the plate surface when compared to those for a laminar boundary layer (Fig. 10.2).

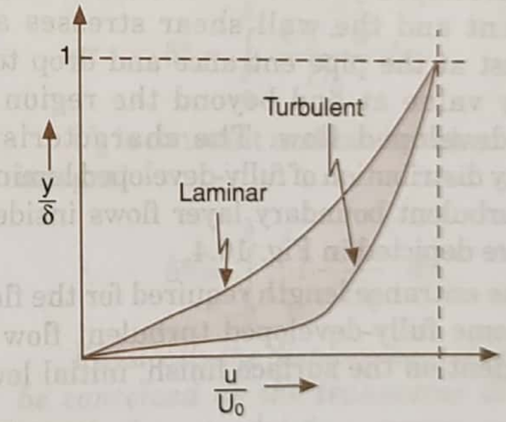

Fig. 10.2. Velocity distribution in laminar and turbulent boundary layers on a flat plate

(*ix*) Velocity gradient and hence the shear stress has a higher value at the plate surface. For a laminar boundary layer the velocity gradient becomes smaller along the flow direction and so does the shear stress. However, for a turbulent boundary layer the shear stress at the plate surface again takes up a high value consistent with the steeper velocity gradient. Figure 10.3 depicts the shear stress distribution for the boundary layer developing on either side of the plate.

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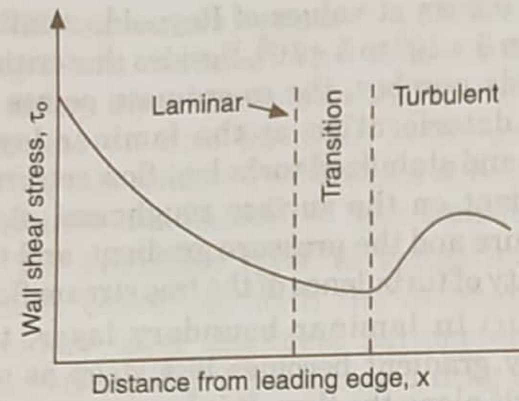

Fig. 10.3. Shear stress distribution on either side of plate

(x) Development of boundary layer for pipe flow proceeds in a fashion similar to that for flow along a flat plate. However, thickness of the boundary layer is limited to the pipe radius because of the flow being within a confined passage. Boundary layers from the pipe walls meet at centre of the pipe and the entire flow acquires the characteristics of a boundary layer. Beyond this point, the velocity profile does not change and it is said to constitute a *fully-developed flow*. Further, the velocity gradient and the wall shear stresses are greatest at the pipe entrance and drop to a steady value at and beyond the region of fully-developed flow. The characteristic velocity distribution of fully-developed laminar and turbulent boundary layer flows inside a pipe are depicted in Fig. 10.4.

The entrance length required for the flow to become fully-developed turbulent flow is dependent on the surface finish, initial level of turbulence, downstream conditions, fluid properties and is generally estimated to be 50 – 80 times the pipe diameter.

10.2. BOUNDARY LAYER PARAMETERS

10.2.1. Boundary layer thickness (δ)

The velocity within a boundary layer approaches the free stream value asymptotically, and so the outer limit of the boundary layer is not easily defined.

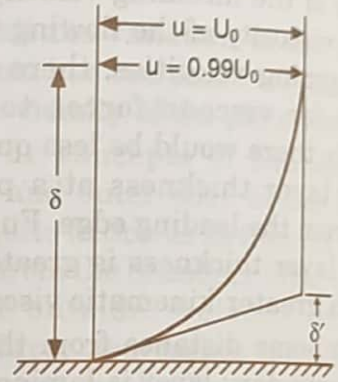

Fig. 10.5. Concept of boundary layer thickness

Usually a distance δ (delta) is prescribed at which the velocity is within 1 percent of its asymptotic value, *i.e.*, $u = 0.99\ U_0$. Then δ becomes a nominal measure of the boundary layer thickness; a measure of the thickness of a region in which major portion of the velocity distribution takes place. Another measure of the boundary layer thickness is determined by finding the intersection of the asymptotic and the tangent to the velocity

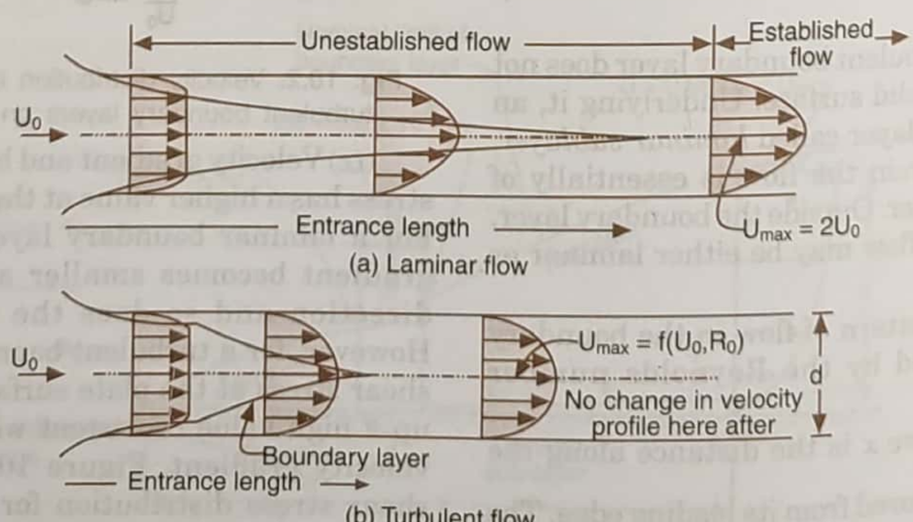

Fig. 10.4. Boundary layer growth in a pipe

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profile drawn at the origin. Evidently, these two concepts give different results. Further, these are devoid of any physical interpretation.

10.2.2. Displacement thickness (δ^*)

Consider an elementary strip (Fig. 10.6) of thickness dy and at instance y from the plate surface. At this elemental strip, the flow velocity can be presumed to have a constant value of u.

Area of elemental strip,

$$dA = b \times dy$$

where b is the width of plate perpendicular to the plane of this page.

Mass flow rate through this strip

$$= \rho \times \text{flow velocity} \times \text{area}$$
$$= \rho\, u\, b\, dy$$

In the absence of plate, the fluid would have moved with a constant velocity equal to the free stream value U_0. The corresponding mass flow rate through this strip would have been

$$= \rho\, U_0\, b\, by$$

$\therefore$ Loss in mass flow rate through the elemental strip is

$$= \rho\, U_0\, b\, dy - \rho\, u\, b\, dy$$
$$= \rho\,(U_0 - u)\, b\, dy$$

Total loss in mass flow rate

$$= \int_0^\delta \rho\,(U_0 - u)\, b\, dy$$

where δ is the value of y at which $u = U_0$

The displacement thickness δ^* is defined as

The thickness of flow (transverse distance measured perpendicular to the boundary of the solid surface) moving at the free stream velocity and having flow rate equal to the loss in flow rate on account of the boundary layer formation.

Mass flow rate through distance δ^*

$$= \rho \times \text{velocity} \times \text{area} = \rho\, U_0\, b\, \delta^*$$

$$\therefore \rho\, U_0\, b\, \delta^* = \int_0^\delta \rho\,(U_0 - u)\, b\, dy$$

$$= \rho\, b \int_0^\delta (U_0 - u)\, dy$$

($\because$ ρ is constant for incompressible flow)

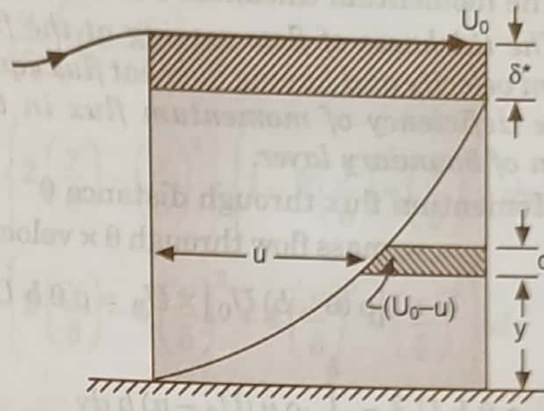

Fig. 10.6. Velocity defect and displacement thickness

Cancelling $\rho\, b$ from both sides, we get

$$U_0\, \delta^* = \int_0^\delta (U_0 - u)\, dy$$

$$\text{or} \qquad \delta^* = \int_0^\delta \frac{U_0 - u}{U_0}\, dy$$

U_0 is constant and can be taken inside the integral

$$\therefore \qquad \delta^* = \int_0^\delta \left(1 - \frac{u}{U_0}\right) dy \qquad ...(10.1)$$

Physically the displacement thickness may be conceived as the transverse distance by which the external free stream is effectivey displaced due to the formation of boundary layer.

10.2.3. Momentum thickness (θ)

For the elementary strip shown in Fig. 10.6

Loss in mass flow rate due to velocity defect

$$= \rho\,(U_0 - u)\, b\, dy$$

Loss in momentum

$$= \text{loss in mass} \times \text{velocity of flow}$$
$$= \rho(U_0 - u)\, b\, dy \times u$$

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∴ Total loss in momentum

$$= \int_0^\delta \rho u (U_0 - u)\, b\, dy$$

The momentum thickness θ is defined as *The thickness of flow moving at the free stream velocity and having moment flux equal to the deficiency of momentum flux in the region of boundary layer.*

Momentum flux through distance θ

$$= \text{mass flow through } \theta \times \text{velocity}$$

$$= [\rho (\theta \times b) U_0] \times U_0 = \rho \theta b U_0^2$$

$$\therefore \rho \theta b U_0^2 = \int_0^\delta \rho u (U_0 - u)\, b\, dy$$

$$= \rho b \int_0^\delta u (U_0 - u)\, dy$$

Cancelling ρ b from both sides, we get :

$$U_0^2 \theta = \int_0^\delta u (U_0 - u)\, dy$$

or
$$\theta = \int_0^\delta \frac{u (U_0 - u)}{U_0^2}\, dy$$

$$= \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy \qquad ...(10.2)$$

Physical the momentum thickness may be conceived as the transverse distance by which the boundary should be displaced to compensate for the reduction in momentum of the flowing fluid on account of boundary layer formation.

10.2.4. Energy thickness (δ**)

For the elementary strip shown in Fig. 10.6.

Mass of fluid = ρ u b dy

Kinetic energy of this fluid

$$= \frac{1}{2} (\text{mass}) \times (\text{velocity})^2$$

$$= \frac{1}{2} (\rho u b\, dy)\, u^2$$

Kinetic energy of this fluid in the absence of boundary layer

$$= \frac{1}{2} (\rho u b\, dy)\, U_0^2$$

Loss in kinetic energy through elementary strip

$$= \frac{1}{2} \rho u b\, dy\, (U_0^2 - u^2)$$

Total loss of kinetic energy

$$= \frac{1}{2} \int_0^\delta \rho u b (U_0^2 - u^2)\, dy$$

*The energy thickness δ** is defined as the thickness of flow moving at the free stream velocity and having the energy equal to deficiency of energy in the boundary layer region.*

Kinetic energy through distance δ**

$$= \frac{1}{2} (\rho b \delta^{**} U_0) \times U_0^2$$

$$= \frac{1}{2} \rho b \delta^{**} U_0^3$$

$$\therefore \frac{1}{2} \rho b \delta^{**} U_0^3 = \frac{1}{2} \int_0^\delta \rho u b (U_0^2 - u^2)\, dy$$

$$= \frac{1}{2} \rho b \int_0^\delta u (U_0^2 - u^2)\, dy$$

Cancelling $\frac{1}{2}$ ρb from both sides, we get

$$\delta^{**} U_0^3 = \int_0^\delta u (U_0^2 - u^2)\, dy$$

or
$$\delta^{**} = \int_0^\delta \frac{u (U_0^2 - u^2)}{U_0^3}\, dy$$

$$= \int_0^\delta \frac{u}{U_0}\left[1 - \left(\frac{u}{U_0}\right)^2\right] dy \qquad ...(10.3)$$

Physically the energy thickness may be conceived as the transverse distance by which

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the boundary layer should be displaced to compensate for the reduction in energy of the flowing fluid on account of the boundary layer formation.

The mass, momentum and energy defect (deficiencies) occur because the streamlines are effectively displaced outwards due to flow retardation near the solid surface.

Shape factor (H) represents the ratio of displacement thickness to momentum thickness.

$$H = \frac{\delta^*}{\theta} \quad ...(10.4)$$

Skin friction coefficient (C_f) refers to the ratio of the local wall shear stress τ_0 to the dynamic pressure of the uniform flow stream,

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_0^2} \quad ...(10.5)$$

EXAMPLE 10.1

Determine the displacement thickness and momentum thickness in terms of the nominal boundary layer thickness δ in respect of the following velocity profiles in the boundary layer on a flat plate.

(i) $\dfrac{u}{U_0} = 2\left(\dfrac{y}{\delta}\right) - \left(\dfrac{y}{\delta}\right)^2$

(ii) $\dfrac{u}{U_0} = \left(\dfrac{y}{\delta}\right)^{1/m}$

where u is the velocity at a height y above the surface and U_0 is the free stream velocity.

Solution : (*i*) Displacement thickness, δ^*

$$= \int_0^\delta \left(1 - \frac{u}{U_0}\right) dy$$

$$= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy$$

$$= \left| y - \frac{2}{\delta}\left(\frac{y^2}{2}\right) + \frac{1}{\delta^2}\left(\frac{y^3}{3}\right)\right|_0^\delta$$

$$= \delta - \frac{2}{\delta}\left(\frac{\delta^2}{2}\right) + \frac{1}{\delta^2}\left(\frac{\delta^3}{3}\right) = \frac{\delta}{3}$$

Momentum thickness, θ

$$= \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy$$

$$= \int_0^\delta \left\{2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right\}\left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy$$

$$= \int_0^\delta \left\{2\left(\frac{y}{\delta}\right) - 5\left(\frac{y}{\delta}\right)^2 + 4\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right\} dy$$

$$= \left|\frac{2}{\delta}\left(\frac{y^2}{2}\right) - \frac{5}{\delta^2}\left(\frac{y^3}{3}\right) + \frac{4}{\delta^3}\left(\frac{y^4}{4}\right) - \frac{1}{\delta^4}\left(\frac{y^5}{5}\right)\right|_0^\delta$$

$$= \frac{2}{\delta}\left(\frac{\delta^2}{2}\right) - \frac{5}{\delta^2}\left(\frac{\delta^3}{3}\right) + \frac{4}{\delta^3}\left(\frac{\delta^4}{4}\right) - \frac{1}{\delta^4}\left(\frac{\delta^5}{5}\right)$$

$$= \delta\left(1 - \frac{5}{3} + 1 - \frac{1}{5}\right) = \frac{2}{15}\delta$$

(*ii*) Displacement thickness, δ^*

$$= \int_0^\delta \left(1 - \frac{u}{U_0}\right) dy$$

$$= \int_0^\delta \left\{1 - \left(\frac{y}{\delta}\right)^{1/m}\right\} dy$$

$$= \left| y - \frac{1}{\delta^{1/m}} \times \frac{m}{m+1} y^{\frac{m+1}{m}}\right|_0^\delta$$

$$= \left| \delta - \frac{m}{m+1}\frac{1}{\delta^{1/m}}\delta^{\frac{m+1}{m}}\right|$$

$$= \left(\delta - \frac{m}{m+1}\delta\right) = \delta\left(1 - \frac{m}{m+1}\right) = \frac{m}{m+1}\delta$$

Momentum thickness, θ

$$= \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy$$

$$= \int_0^\delta \left(\frac{y}{\delta}\right)^{1/m}\left\{1 - \left(\frac{y}{\delta}\right)^{1/m}\right\} dy$$

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