Determine the inverse z-Transform of the

$$F(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

$$= \frac{1}{\left(1+\frac{1}{z}\right)\left(1-\frac{1}{z}\right)^2}$$

$$= \frac{1}{\left(\frac{z+1}{z}\right)\left(\frac{z-1}{z}\right)^2} = \frac{1}{\frac{(z+1)(z-1)^2}{z^3}}$$

$$= \frac{1}{(z+1)(z-1)^2} \times \frac{z^3}{1} = \frac{z^3}{(z+1)(z-1)^2}$$

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$$\frac{F(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

By Partial fraction expansion, we can write.

$$\frac{F(z)}{z} = \frac{A_1}{(z+1)} + \frac{A_2}{(z-1)^2} + \frac{A_3}{(z-1)}$$

$$A_1 = (z+1) \times \frac{F(z)}{z} = \cancel{(z+1)} \times \frac{z^2}{\cancel{(z+1)}(z-1)^2}\bigg|_{z=-1} = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4} = 0.25$$

$$A_2 = (z-1)^2 \times \frac{F(z)}{z} = \cancel{(z-1)^2} \times \frac{z^2}{(z+1)\cancel{(z-1)^2}}\bigg|_{z=1} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$A_3 = \frac{d}{dz}\left[(z-1)^2 \frac{F(z)}{z}\right]_{z=1}$$

$$A_3 = \frac{d}{dz}\left[\cancel{(z-1)^2} \times \frac{z^2}{(z+1)\cancel{(z-1)^2}}\right]_{z=1}$$

$$A_3 = \frac{d}{dz}\left[\frac{z^2}{z+1}\right]_{z=1} = \frac{(z+1)\times 2z - z^2}{(z+1)^2}\bigg|_{z=1} = \frac{(1+1)\times 2\times 1 - 1}{(1+1)^2} = \frac{4-1}{4} = \frac{3}{4} = 0.75$$

$$\therefore \frac{F(z)}{z} = \frac{A_1}{(z+1)} + \frac{A_2}{(z-1)^2} + \frac{A_3}{(z-1)}$$

$$= \frac{0.25}{z+1} + \frac{0.5}{(z-1)^2} + \frac{0.75}{(z-1)}$$

$$F(z) = 0.25 \cdot \frac{z}{z+1} + 0.5 \cdot \frac{z}{(z-1)^2} + 0.75 \cdot \frac{z}{(z-1)}$$

We know that

$$Z[a^k] = \frac{z}{z-a} \; ; \; z\left\{\frac{az}{(z-a)^2}\right\} = ka^k \text{ and } z\{u(k)\} = \frac{z}{z-1}$$

taking Inverse z-Transform of $F(z)$ we get

$$f(k) = 0.25 \cdot \frac{z}{z-(-1)} + 0.5 \cdot \frac{z}{(z-1)^2} + 0.75 \cdot \frac{z}{z-1}$$

$$f(k) = 0.25 \cdot (-1)^k + 0.5\, k(1)^k + 0.75\, u(k) \quad \text{Ans}$$

continuous function

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Theorem

(1) $Z[k f(k)] = \left(-z \frac{d}{dz}\right) \cdot F(z)$

$$L.H.S = Z[k f(k)]$$

$$= \sum_{k=0}^{\infty} k f(k) \cdot z^{-k}$$

$$\Rightarrow -z \sum_{k=0}^{\infty} \frac{k f(k) \cdot z^{-k}}{-z}$$ [Multiply & divide by $-z$.]

$$\Rightarrow -z \left[\sum_{k=0}^{\infty} -k f(k) \cdot z^{-k-1}\right]$$

$$\Rightarrow -z \left[\sum_{k=0}^{\infty} f(k) \frac{d}{dz}(z^{-k})\right]$$ $(\cancel{-k z^{-k-1}} \cdot z$

$$\Rightarrow -z \cdot \frac{d}{dz} \left[\sum_{k=0}^{\infty} f(k) z^{-k}\right]$$

$$\boxed{Z[k f(k)] = \left(-z \frac{d}{dz}\right) F(z)}$$ Proof

$$Z[k^n f(k)] = \left(-z \frac{d}{dz}\right)^n \cdot F(z)$$

$$Z[k\, u(k)] = -z \frac{d}{dz} \cdot U(z)$$ Discrete time signal দ্বারা Z transform

$$= -z \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$Z[k u(k)] = \frac{z}{(z-1)^2}$$

$$\frac{(z-1)\frac{d}{dz}(z) - z\frac{d}{dz}(z-1)}{(z-1)^2}$$

$$\frac{(z-1)\times 1 - z\times 1}{(z-1)^2}$$

$$\frac{z-1-z}{(z-1)^2} = \frac{-1}{(z-1)^2} \times z$$

$$= \frac{z}{(z-1)^2}$$

Linearity Theorem

$$Z[a f(k) + b g(k) + c h(k)]$$

$$\Rightarrow a F(z) + b G(z) + c H(z)$$

$$Z[a u(k)] = a \cdot U(z)$$

$$= a \cdot \frac{z}{z-1}$$

For unit step function

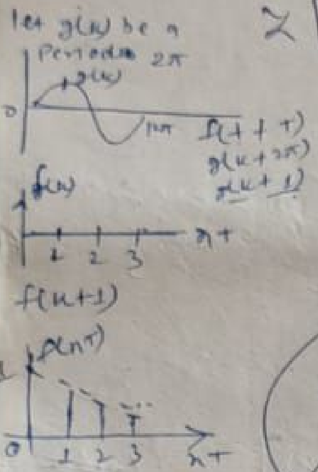
$u(t) = 1$

$Z[u(t)] = U(z) = 1$

$= \frac{z}{z-1}$

8th) Shifting theorem

Let $g(n) = f(k+1)$



$$Z[f(k+1)] = \sum_{k=0}^{\infty} f(k+1)\, z^{-k}$$

$$= z \sum_{k=0}^{\infty} f(k+1)\, z^{-k-1}$$

(such cond. is comes from multipl & divided z)

$$= z \sum_{k=0}^{\infty} f(k+1)\, z^{-(k+1)}$$

let $k+1 = n$

$$= z \sum_{k=0}^{\infty} f(n)\, z^{-n}$$

when $k=0$, $n=1$

$k=\infty$, $n=\infty$.

$$\Rightarrow z \sum_{n=1}^{\infty} f(n)\, z^{-n}$$

$$\Rightarrow z\left[\sum_{n=0}^{\infty} f(n) z^{-n} - f(0) z^{-0}\right]$$

$$\Rightarrow z\left[F(z) - f(0)\right]$$

$$Z[f(k+1)] = z[F(z) - f(0)]$$

$$= zF(z) - zf(0)$$

|||ly, $Z[f(k+2)] = z^2 F(z) - z^2 f(0) - z f(1)$

$$Z[f(k+3)] = z^3 F(z) - z^3 f(0) - z^2 f(1) - z f(2)$$

$$Z[f(k+n)] = z^n F(z) - z^n f(0) - z^{n-1} f(1) - z^{n-2} f(2) - z^{n-3} f(3) -$$

$$Z[f(k-n)] = z^{-n} F(z)$$
$$Z[f(k-1)] = z^{-1} F(z)$$
$$Z[f(k-2)] = z^{-2} F(z)$$

$$\boxed{\mathcal{L}[e^{-at}] = \frac{1}{s+a}}$$
$$\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$a^k \longleftrightarrow \frac{z}{z-a}$$

$a^{k-1} \to \frac{1}{z-a}$

$(-a)^{k-1} \to \frac{1}{z+a}$

$u(k) \to \frac{z}{z-1}$

$\delta(k) \to 1$

$$\frac{1}{(z-a)} \longrightarrow (a)^{k-1}$$

$$\frac{1}{z+a} \longrightarrow (-a)^{k-1}$$

$$\frac{z}{(z-1)} \xrightarrow{\text{inverse}} u(k)$$

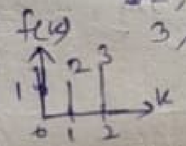
$$1 \xrightarrow{\text{inverse}} \delta(k)$$

Shifting in time

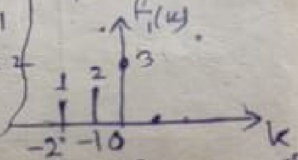
A signal $f(k)$ may be ~~shiftings~~ shifted in time by replacing the independent variable k by $(k-m)$ where m is an integer. If m is a +ve integer, the time shift results in a delay by m units of time. If ~~m~~ m is a −ve integer the time shift results in an advance of the signal by m unit in time. The delay result in shifting each sample of $f(k)$ to right. Advance results in shifting each sample of $f(k)$ to left.

example

let $f(k) = 1,\ k=0$
$= 2,\ k=1$
$= 3,\ k=2$

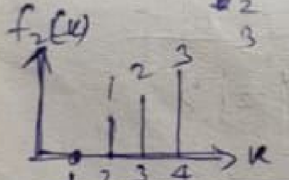


$f_1(k) = f(k+2) =$



$f_1(k) = f(k+2) = 1,\ k=-2$
$= 2\ \ k=-1$
$= 3\ \ k=0$

$f_2(k) = f(k-2) = 1\ \ k=2$
$= 2\ \ k=3$
$= 3\ \ k=4$



Some important z-transform pairs and corresponding ROC's

ROC's → Region of convergence

S.No	Discrete-time signal $x(n)$	z-transform $X(z)$	ROC
1.	$\delta(n)$	1	All z-plane
2.	$\delta(n-k)$	z^{-k}	$\lvert z\rvert > 0,\ k>0$ $\lvert z\rvert < \infty,\ k<0$
3.	$u(n)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$
4.	$n u(n)$	$\frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5.	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$

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S.No	discrete time signal x(n)	Z-transform X(z)	ROC
6.	$n a^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
7.	e^{-an}	$\frac{z}{z-e^{-a}}$	$\|z\| > \|e^{-a}\|$
8.	$n e^{-an}$	$\frac{z e^{-a}}{(z-e^{-a})^2}$	$\|z\| > \|e^{-a}\|$
9.	$\sin w_0 n$	$\frac{z \sin w_0}{z^2 - 2z\cos w_0 + 1}$	$\|z\| > 1$
10.	$\cos w_0 n$	$\frac{z(z-\cos w_0)}{z^2 - 2z\cos w_0 + 1}$	$\|z\| > 1$
11.	$\sinh w_0 n$	$\frac{z \sinh w_0}{z^2 - 2z\cosh w_0 + 1}$	$\|z\| > 1$
12.	$\cosh w_0 n$	$\frac{z(z-\cosh w_0)}{z^2 - 2z\cosh w_0 + 1}$	$\|z\| > 1$
13.	$e^{-an}\sin w_0 n$	$\frac{z e^{-a}\sin w_0}{z^2 - 2z e^{-a}\cos w_0 + e^{-2a}}$	$\|z\| > \|e^{-a}\|$
14.	$e^{-an}\cos w_0 n$	$\frac{z(z-e^{-a}\cos w_0)}{z^2 - 2z e^{-a}\cos w_0 + e^{-2a}}$	$\|z\| > \|e^{-a}$
15.	$a^n \sin w_0 n$	$\frac{z a \sin w_0}{z^2 - 2za\cos w_0 + a^2}$	$\|z\| > \|a\|$.
16.	$a^n \cos w_0 n$	$\frac{z(z-a\cos w_0)}{z^2 - 2za\cos w_0 + a^2}$	$\|z\| > \|a\|$

(2) inverse Z-Transform

(1) Partial fraction (2) Long division Method (3) Residue Method