Lecture No. 7

-STRAIN ENERGY-

7-1 Introduction: -

Strain energy is as the energy which is stored within a material when work has been done on the material. Here it is assumed that the material remains elastic whilst work is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

Strain energy U = work done

Thus for a gradually applied load the work done in straining the material will be given by the shaded area under the load-extension graph of Fig.

$$U = \frac{1}{2} P \delta$$

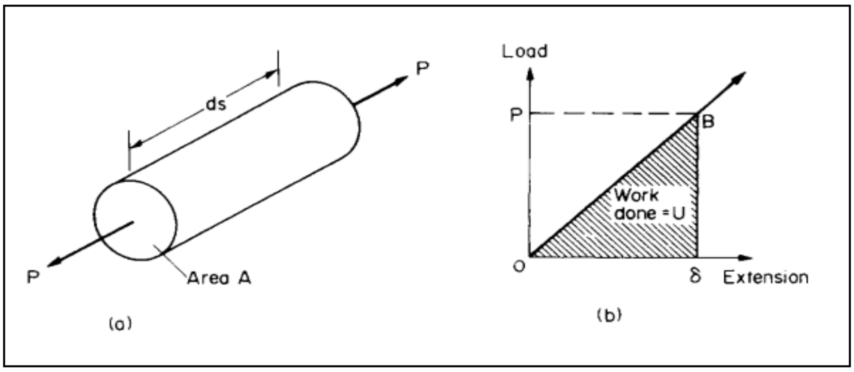


Figure 7.1: - Work done by a gradually applied load.

The unshaded area above the line OB of Fig. 7.1 is called the complementary energy, a quantity which is utilized in some advanced energy methods of solution and is not considered within the terms of reference of this text.

7-2 Strain energy - tension or compression: -

a- Neglecting the weight of the bar: -

Consider a small element of a bar, length ds, shown in Fig. 7.1. If a graph is drawn of load against elastic extension the shaded area under the graph gives the work done and hence the strain energy,

$U = \frac{1}{2} P \delta$ ....(1)

But young modulus $E = \frac{Pds}{A\delta}$ $\therefore \ \delta = \frac{Pds}{AE}$ .... (2)

Now, substituting eqn. (2) in (1)

For bar element, $U = \frac{P^2 ds}{2AE}$

$\therefore$ Total strain energy for a bar of length L, $U = \int_o^L (\frac{P^2 ds}{2AE})$

$$\boxed{U = \frac{P^2 L}{2AE}} \qquad \textbf{....7.1}$$

b- Including the weight of the bar: -

Consider now a bar of length L mounted vertically, as shown in Fig. 7.2. At any section A B the total load on the section will be the external load P together with the weight of the bar material below AB.

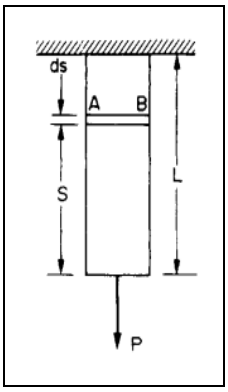


Figure 7.2: - Direct load - tension or compression.

Load on section *A B* = $P \pm \rho gAs$

The positive sign being used when *P* is tensile and the negative sign when *P* is compressive. Thus, for a tensile force *P* the extension of the element *ds* is given by the definition of Young's modulus *E* to be

$$\delta = \frac{\sigma ds}{E}$$

$$\delta = \frac{(P \pm \rho gAs)ds}{AE}$$

But work done = $\frac{1}{2}$ load x extension

$$= \frac{1}{2}(P \pm \rho gAs)\frac{(P \pm \rho gAs)ds}{AE}$$

$$= \frac{P^2}{2AE}ds + \frac{P\rho g}{E}sds + \frac{(\rho g)^2 A}{2E}s^2 ds$$

So, the total strain energy or work done is,

$$= \int_0^L \frac{P^2}{2AE}ds + \int_0^L \frac{P\rho g}{E}sds + \int_0^L \frac{(\rho g)^2 A}{2E}s^2 ds$$

$$\boxed{U = \frac{P^2 L}{2AE} + \frac{P\rho gL^2}{2E} + \frac{(\rho g)^2 AL^3}{6E}} \qquad \text{....7.2}$$

7-2 Strain energy-shear: -

Consider the elemental bar now subjected to a shear load Q at one end causing deformation through the angle y (the shear strain) and a shear deflection 6, as shown in Fig. 7.3.

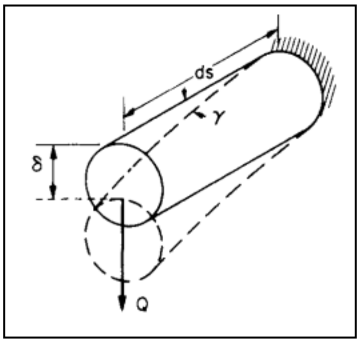


Figure 7.3: - Shear.

Strain energy U = work done = $\frac{1}{2}Q\delta = \frac{1}{2}Q\gamma ds$ …. (1)

But modulus of rigidity $G = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$

$$\gamma = \frac{Q}{AG} \qquad \text{…. (2)}$$

Substitute eqn. (2) in (1),

$$U = \frac{1}{2}Q\frac{Q}{AG}ds$$

Shear strain energy$= \frac{Q^2}{2AG}ds$

:. Total strain energy resulting from shear$= \int_0^L \frac{Q^2}{2AG}ds$

$$\boxed{U = \frac{Q^2 L}{2AG}}$$

7-3 Strain energy –bending: -

Let the element now be subjected to a constant bending moment M causing it to bend into an arc of radius R and subtending an angle $d\Theta$ at the center (Fig. 7.4). The beam will also have moved through an angle $d\Theta$.

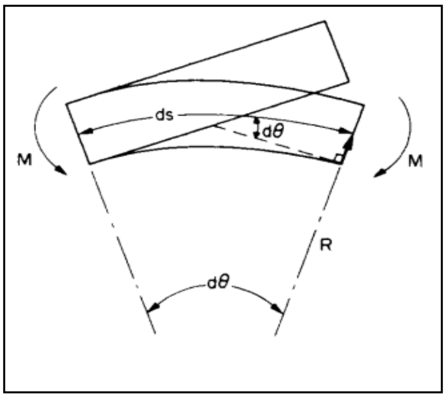


Figure 7.4: - Bending.

Strain energy = work done = $\frac{1}{2}$ moment x angle turned through (in radians)

$$= \frac{1}{2} M d\theta \qquad \ldots. (1)$$

But $ds = Rd\theta$ and $\frac{M}{I} = \frac{E}{R}$

$$\therefore d\theta = \frac{ds}{R} = \frac{M}{EI} ds \qquad \ldots. (2)$$

Substitute eqn. (2) in (1),

∴ Strain energy = $\frac{1}{2} M \frac{M}{EI} ds = \frac{M^2 ds}{2EI}$

Total strain energy resulting from bending, $U = \int_0^L \frac{M^2 ds}{2EI}$

$$\therefore \boxed{U = \frac{M^2 L}{2EI}}$$

7-4 Strain energy – torsion: -

The element is now considered subjected to a torque T as shown in Fig. 7.5, producing an angle of twist $d\Theta$ radians.

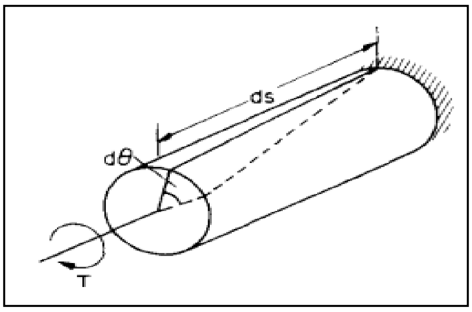


Figure 7.5: - Torsion.

Strain energy = work done = $\frac{1}{2}$T $d\Theta$ …. (1)

But, from the simple torsion theory,

$$\frac{T}{J} = \frac{Gd\theta}{ds} \text{ and } d\theta = \frac{Tds}{GJ} \qquad \ldots. (2)$$

Substitute eqn. (2) in (1),

∴ Total strain energy resulting from torsion,

$$U = \int_0^L \frac{T^2}{2GJ} ds \rightarrow \boxed{U = \frac{T^2 L}{2GJ}}$$

<u>Note:</u> - It should be noted that in the four types of loading case considered above the strain energy expressions are all identical in form,

$$\boxed{U = \frac{(unit\ applied\ load)^2 \times L}{2 \times Product\ of\ two\ related\ constants}}$$

7-5 Castigliano's first theorem assumption for deflection: -

If the total strain energy of a body or framework is expressed in terms of the external loads and is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load,

i.e. $a = \frac{\delta U}{\delta P_a}$, $b = \frac{\delta U}{\delta P_b}$ and $c = \frac{\delta U}{\delta P_c}$

Where a, b and c are deflections of a beam under loads P_a, P_b and P_c etc. as shown in fig 7.6.

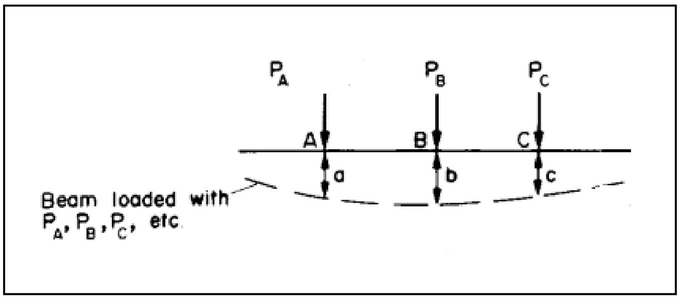


Figure 7.6: - Any beam or structure subjected to a system of applied concentrated loads P_a, P_b, P_c, . . . P_n,, etc.

In most beam applications the strain energy, and hence the deflection, resulting from end loads and shear forces are taken to be negligible in comparison with the strain energy resulting from bending (torsion not normally being present),

$$U = \int \frac{M^2 ds}{2EI}$$

$$\therefore \frac{\partial U}{\partial P} = \frac{\partial U}{\partial M}\frac{\partial M}{\partial P}$$

$$\therefore \frac{\partial U}{\partial P} = \int \frac{2Mds}{2EI}\frac{\partial M}{\partial P} \quad \rightarrow \quad \boxed{\delta = \frac{\partial U}{\partial P} = \int \frac{M}{EI}\frac{\partial M}{\partial P} ds}$$

7-6 Application of Castigliano's theorem to angular movements:

If the total strain energy, expressed in terms of the external moments, be partially differentiated with respect to one of the moments, the result is the angular deflection (in radians) of the point of application of that moment and in its direction,

$$\boxed{\theta = \int \frac{M}{EI}\frac{\partial M}{\partial M_i} ds}$$

Example 7-1: -Determine the diameter of an aluminum shaft which is designed to store the same amount of strain energy per unit volume as a 50mm diameter steel shaft of the same length. Both shafts are subjected to equal compressive axial loads. What will be the ratio of the stresses set up in the two shafts?

$E_{steel} = 200$ GN/m^2; $E_{aluminum} = 67$ GN/m^2.

Sol.

Strain energy per unit volume $= \dfrac{P^2}{2A^2E}$ but $P = \sigma A$

Strain energy per unit volume $U = \dfrac{\sigma^2}{2E}$

Since the strain energy/unit volume in the two shafts is equal,

$$\frac{{\sigma_A}^2}{2E_A} = \frac{{\sigma_S}^2}{2E_S}$$

$\frac{{\sigma_A}^2}{{\sigma_S}^2} = \frac{E_A}{E_S} = \frac{67}{200} = \frac{1}{3}$ approximately.

$3{\sigma_A}^2 = {\sigma_S}^2 \rightarrow \frac{\sigma_A}{\sigma_S} = \sqrt{\frac{1}{3}}$

$3\left(\frac{P_A}{A_A}\right)^2 = \left(\frac{P_S}{A_S}\right)^2$ but P_s=P_A=P and $A = \frac{\pi}{4}D^2$

$3\left(\frac{1}{{D_A}^2}\right)^2 = \left(\frac{1}{{D_S}^2}\right)^2 \rightarrow 3{D_S}^4 = {D_A}^4 \rightarrow D_A = \sqrt[4]{3}D_S$

$$\therefore\ D_A = \sqrt[4]{3}(0.050) = 0.0658\,m$$

Example 7-2: - Two shafts are of the same material, length and weight. One is solid and 100 mm diameter, the other is hollow. If the hollow shaft is to store 25 % more energy than the solid shaft when transmitting torque, what must be its internal and external diameters? Assume the same maximum shear stress applies to both shafts.

Sol.

Let A be the solid shaft and B the hollow shaft. If they are the same weight and the same material their volume must be equal.

$$\frac{\pi}{4}{D_A}^2 = \frac{\pi}{4}\left({D_B}^2 - {d_B}^2\right)$$

$${D_A}^2 = \left({D_B}^2 - {d_B}^2\right) = (0.100^2) = 0.1\,m^2 \qquad \text{.... (1)}$$

Now for the same maximum shear stress,

$$\tau = \frac{TD}{2J}$$

$$\frac{T_A D_A}{2J_A} = \frac{T_B D_B}{2J_B} \rightarrow \frac{T_A}{T_B} = \frac{D_B J_A}{D_A J_B} \qquad \text{.... (2)}$$

But the strain energy of B = 1.25 x strain energy of A.

$$U = \frac{T^2 L}{2GJ}$$

$$\frac{T_B{}^2 L}{2GJ_B} = 1.25\frac{T_A{}^2 L}{2GJ_A} \rightarrow \frac{T_A{}^2}{T_B{}^2} = \frac{J_A}{1.25J_B} \qquad \text{.... (3)}$$

Now substitute eqn. (2) in (3),

$$\frac{D_B{}^2}{D_A{}^2} = \frac{J_B}{1.25J_A} \rightarrow \frac{D_B{}^2}{D_A{}^2} = \frac{D_B{}^4 - d_B{}^4}{1.25D_A{}^4}$$

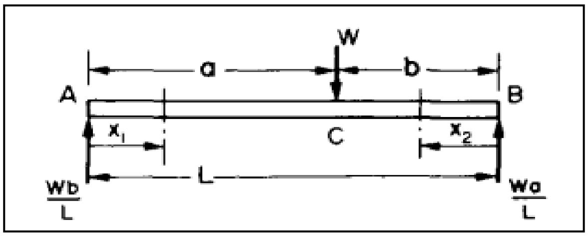
$$\therefore D_B{}^2 = \frac{D_B{}^4 - d_B{}^4}{1.25D_A{}^2} \rightarrow \frac{D_B{}^4 - (D_B{}^2 - 0.1)^2}{1.25D_A{}^2}$$

$\therefore D_B = 115.47\ mm$ Substitute in eqn. (1)

$$\therefore d_B = 57.74\ mm.$$

Example 7-3: - Using Castigliano's first theorem, obtain the expressions for (a) the deflection under a single concentrated load applied to a simply supported beam as shown in Figure below, (b) the deflection at the center of a simply supported beam carrying a uniformly distributed load.

a-



$$\delta = \int_B^A \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

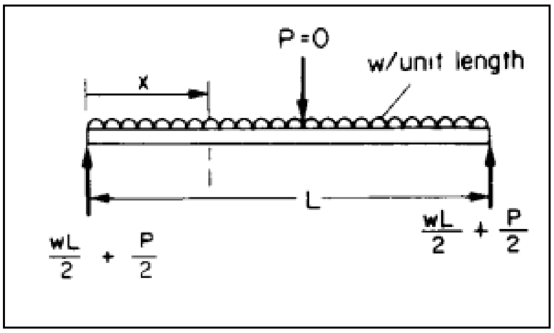
$$= \int_A^C \frac{M}{EI}\frac{\partial M}{\partial W}\,ds + \int_C^B \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

$$= \frac{1}{EI}\int_0^a \frac{Wbx_1}{L}\times\frac{bx_1}{L}\times dx_1 + \frac{1}{EI}\int_0^b \frac{Wax_2}{L}\times\frac{ax_2}{L}\times dx_2$$

$$= \frac{Wb^2}{L^2EI}\int_0^a x_1^2\,dx_1 + \frac{Wa^2}{L^2EI}\int_0^b x_2^2\,dx_2$$

$$= \frac{Wb^2a^3}{3L^2EI} + \frac{Wa^2b^3}{3L^2EI} = \frac{Wa^2b^2}{3L^2EI}(a+b) = \boldsymbol{\frac{Wa^2b^2}{3LEI}}$$

b-



$$\delta = \int_0^L \frac{Mm}{EI}\,ds = 2\int_0^{L/2} \frac{Mm}{EI}\,ds$$

$$M = \frac{wL}{2} - \frac{wx^2}{2} \quad \text{and} \quad m = \frac{x}{2}$$

$$\delta = \frac{2}{EI}\int_0^{L/2}\left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{x}{2}\,dx$$

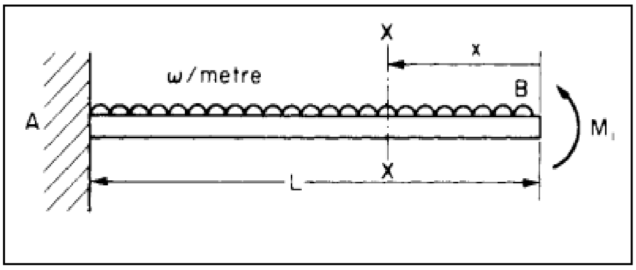
$$= \frac{1}{2EI}\int_0^{L/2}(wLx^2 - wx^3)\,dx$$

$$\delta = \frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/2}$$

$$= \frac{wL^4}{2EI}\left[\frac{1}{24} - \frac{1}{64}\right]$$

$$= \frac{wL^4}{2EI}\left[\frac{8-3}{192}\right] = \mathbf{\frac{5WL^4}{384EI}}$$

Example 7-4: - Derive the equation for the slope at the free end of a cantilever carrying a uniformly distributed load over its full length.



Sol.

$$M = M_i - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial M_i} = 1$$

$$\theta = \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial M_i} \cdot dx$$

$$= \frac{1}{EI}\int_0^L \left(M_i - \frac{wx^2}{2}\right)(1)\,dx$$

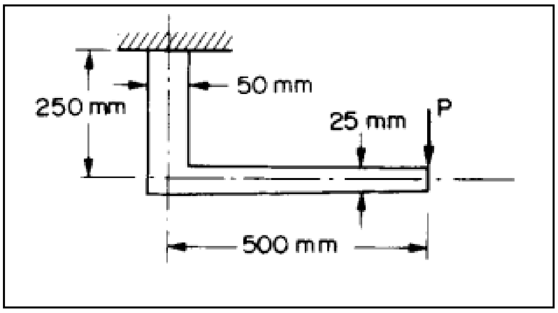
$$\theta = \frac{-w}{2EI}\int_0^L x^2 \,.\, dx = \frac{wL^3}{6EI} \text{ radian}$$

Example 7-5: -Determine, for the cranked member shown in Figure below:

(a) The magnitude of the force P necessary to produce a vertical movement of P of 25 mm;

(b) The angle, in degrees, by which the tip of the member diverges when the force P is applied.

The member has a uniform width of 50mm throughout. E = 200GN/m^2.



Sol. (a)

Horizontal beam: -

$$M = Px, \frac{\partial M}{\partial P} = x$$

$$\delta = \int \frac{M}{EI}\frac{\delta M}{\delta P}dx$$

$$I_h = \frac{bh^3}{12} = \frac{0.05(0.025)^3}{12} \rightarrow \frac{1}{(EI)_h} = 76.8 * 10^{-6}$$

$$\delta_{horizontal} = 76.9 * 10^{-6} \int_0^{0.5} Px^2 dx$$

$$\delta_{horizontal} = 76.9 * 10^{-6} P \left[\frac{x^3}{3}\right]_0^{0.5} = 3.2 * 10^{-6} P$$

Vertical beam: -

$$M = 0.5P, \frac{\partial M}{\partial P} = 0.5$$

$$I_v = \frac{bh^3}{12} = \frac{0.05(0.05)^3}{12} \rightarrow \frac{1}{(EI)_v} = 9.6 * 10^{-6}$$

$$\delta_{verticlal} = 9.6 * 10^{-6} \int_0^{0.25} 0.5P(0.5)dx$$

$\delta_{verticlal} = 9.6 * 10^{-6}[0.25Px]_0^{0.25} = 600 * 10^{-9}P$

$\delta_{total} = \delta_{horizontal} + \delta_{verticlal}$

$0.025 = 3.2 * 10^{-6}P + 600 * 10^{-6}P$

P = 6.58kN.

Sol. (b)

$\theta = \int \frac{M}{EI}\frac{\delta M}{\delta M_i}dx$

Horizontal beam: -

$M = Px + M_i \, , \frac{\partial M}{\partial M_i} = 1$

$\theta_{horizontal} = 76.9 * 10^{-6} \int_0^{0.5}(Px + M_i)(1)dx$

$\theta_{horizontal} = 76.9 * 10^{-6} \left[\frac{Px^2}{2} + M_i x\right]_0^{0.5} = 76.9 * 10^{-6} \frac{0.5^2P}{2}$ when $M_i = 0$

But P=6.58kN.

$\theta_{horizontal} = 63.168 * 10^{-3} rad.$

Vertical beam: -

$M = 0.5P + M_i \, , \frac{\partial M}{\partial M_i} = 1$

$\theta_{vertical} = 9.6 * 10^{-6} \int_0^{0.25}(0.5P + M_i)(1)dx$

$\theta_{vertical} = 9.6 * 10^{-6}[0.5Px + M_i x]_0^{0.25}$ but $M_i = 0$ and P=6.58kN.

$\theta_{vertical}$ =7.896*10^{-3} $rad.$

$\theta_{total} = \theta_{horizontal} + \theta_{verticlal}$

$\theta_{total} = 63.168 * 10^{-3}$ +7.896*10^{-3}=0.071 rad=4.1°

<u>H.W.</u> A semicircular frame of flexural rigidity (**EI**) is built in at **A** and carries a vertical load **W** at **B** as shown in Figure (1). Calculate the magnitude of horizontal deflection at **B**.

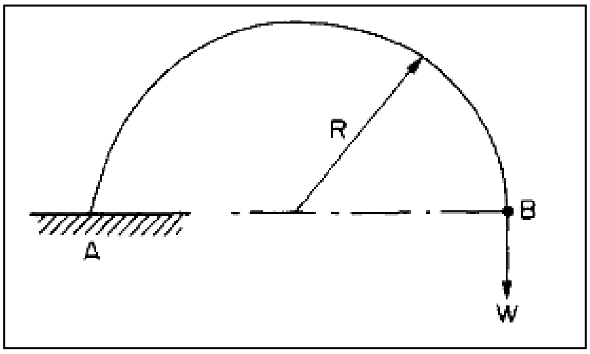


Figure (1)

..…………………….End…………………………