(6) (a) If $z\{x(n)\} = X(z)$, Then Prove that

$$z\{n\,x(n)\} = z \frac{dX(z)}{dz}$$

Soln. We have

$$z\{x(n)\} = X(z)$$

$$\text{i.e. } X(z) = \sum_{n=0}^{\infty} x(n)\cdot z^{-n} \quad -(1)$$

Differentiating w.r.t z, we get

$$\frac{d}{dz}X(z) = -\sum_{n=0}^{\infty} n\,x[n]\,z^{-n-1} \quad -(2)$$

Now, we have

$$z[n\,x(n)] = \sum_{n=0}^{\infty} n\,x[n]\,z^{-n}$$

$$= \sum_{n=0}^{\infty} n\,\underline{x[n]\,.\,z\,.\,z^{-n-1}} \longrightarrow$$ z से Multiply & divided

$$= -z\left[-\sum_{n=0}^{\infty} n\,x[n]\,.\,z^{-n-1}\right]$$

$$= -z\frac{d}{dz}X(z) \quad [\text{from } (2)]$$

Hence, $\boxed{z\{n\,x[n]\} = -z\frac{d}{dz}X(z)}$ Prove.

17

(6) (b) A discrete time signal is expressed as

$$x[n] = n^2 u[n]$$

Determine its z-Transform.

Soln. we have

$$x[n] = n^2 u[n] \quad -(1)$$

$$\therefore x[n+1] = (n+1)^2 u[n] \quad -(2)$$

$$x[n+1] - x[n] = \{(n+1)^2 - n^2\}u[n]$$

from (2) to (1)

$$= (2n+1)\,u[n] \quad -(3)$$

(2)

Now, $Z\left[x(n+1)-x[n]\right] = Z\left[X(z)-X(0)\right] - X(z)$

$\because X(0)=0$

$\therefore Z\left[x(n+1)-x(n)\right] = (z-1)X(z)$ — (4)

from (3)

$$Z\left[x(n+1)-x(n)\right] = Z\left[(2n+1)u(n)\right]$$

$$= 2Z\left[nu(n)\right] + Z\left[u(n)\right]$$

$$= \frac{2z}{(z-1)^2} + \frac{z}{z-1}$$

$$= \frac{2z+z(z-1)}{(z-1)^2}$$

$$= \frac{2z+z^2-z}{(z-1)^2}$$

$$\therefore Z\left[x(n+1)-x(n)\right] = \frac{z(2+z-1)}{(z-1)^2}$$

from (4) & (5)

$$= \frac{z(z+1)}{(z-1)^2} \quad \text{— (5)}$$

$Z[\cancel{2n+1}]$

$\cancel{2zn}$

$\cancel{x(z)}$ $(z-1)X(z) = \frac{z(z+1)}{(z-1)^2}$

$$X(z) = \frac{z(z+1)}{(z-1)^3}$$

18

Hence $\boxed{Z\left[x(n)\right] = \frac{z(z+1)}{(z-1)^3}}$

(2) (a) If $Z[f(t)] = F(z)$, then Prove that

$$Z[t f(t)] = -Tz \frac{d}{dz} f(z)$$

Soln. given $Z[f(t)] = F(z)$

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad \text{--- (1)}$$

Now, differentiating w.r.to z

$$\frac{d}{dz} F(z) = \frac{d}{dz} \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \frac{d}{dz} (z^{-n})$$

n का Multiply & divide

$$\frac{d}{dz} F(z) = - \sum_{n=0}^{\infty} n f(nT) z^{-n-1} \quad \text{--- (11)}$$

Now, we have to Prove

$$Z[t f(t)] = -Tz \frac{d}{dz} F(z)$$

Here,

$$Z[t f(t)] = \sum_{n=0}^{\infty} (nT) f(nT) z^{-n}$$ — due to ramp fun^n

$$= T \sum_{n=0}^{\infty} n f(nT) z^{-n}$$ — const. term outside

$$= Tz \sum_{n=0}^{\infty} n f(nT) z^{-n-1}$$ — z से Multiply and divide

$$= -Tz \left[\sum_{n=0}^{\infty} n f(nT) z^{-n-1} \right]$$

19

Now, using (11), we get

$$\boxed{Z[t f(t)] = -Tz \frac{d}{dz} F(z)}$$ Ans

(9) (b) Initial value theorem for Z-Transform.

According to the initial value theorem for Z-Transform

$$\boxed{f(0) = \lim_{s \to \infty} F(z)}$$

20

Proof. We know that

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$F(z) = f(0) + f(T) z^{-1} + f(2T) z^{-2} + \cdots \quad (1)$$

Now, clearly if we put limit $z \to \infty$ on the both ~~right~~ hand side of (1), we will get $f(0)$

Hence

$$\boxed{f(0) = \lim_{s \to \infty} F(z)}$$

Final value theorem of z - Transform method

$$\lim_{k\to\infty} f(k) = f(\infty) = \lim_{z\to 1} (z-1) F(z)$$

Proof, we know that for a causal signal

$$z\left[f(k)\right] = F(z) = \underline{\sum_{k=0}^{\infty} f(k) z^{-k}} \quad \text{—} ①$$

$$z\left[f(k+1)\right] = zF(z) - zf(0) = \underline{\sum_{k=0}^{\infty} f(k+1) z^{-k}} \quad \text{—} ②$$

from eqn ② - eqn ①

$$zF(z) - zf(0) - F(z) = \sum_{k=0}^{\infty} f(k+1) z^{-k} - \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$(z-1) F(z) - zf(0) = \sum_{k=0}^{\infty} \left[f(k+1) - f(k)\right] z^{-k}$$

$$(z-1) F(z) - zf(0) = \left[f(1) - f(0)\right] z^{-0} + \left[f(2) - f(1)\right] z^{-1} + \left[f(3) - f(2)\right] z^{-2} + \cdots$$

Taking limit $z \to 1$ on both sides

$$\lim_{z\to 1} (z-1) F(z) - zf(0) = \cancel{f(1)} - f(0) + \cancel{f(2)} - \cancel{f(1)} + f(3) - \cancel{f(2)} + \cdots + f(\infty) - f(\infty - 1)$$

$$\lim_{z\to 1} (z-1) F(z) - \cancel{zf(0)} = -\cancel{f(0)} + f(\infty)$$

$$\lim_{z\to 1} (z-1) F(z) = f(\infty)$$

$$\boxed{\therefore f(\infty) = \lim_{z\to 1} (z-1) F(z)}$$

2