Q Discuss Schrodinger's Time independent Wave Equation.

Soln :- Let us consider a particle of mass 'm' moving with a velocity 'v' in the form of wave and the associated De-Broglie wave-length is given by $\lambda = h/mv$. Let ψ be the wave fun of the particle along x, y & z coordinate axes at any time 't'.

The classical differential equation of the wave system is given by :-

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad \text{---} (1)$$

The solution of the above equation is given by,

$$\psi(x,y,z,t) = \psi_0(x,y,z) e^{-i\omega t} \quad \text{---} (2)$$

Differentiating eqn (2) w.r.to. 't' we get,

$$\frac{\partial \psi}{\partial t} = -i\omega \psi_0 e^{-i\omega t}$$

and
$$\frac{\partial^2 \psi}{\partial t^2} = (-i\omega)(-i\omega)\psi_0 e^{-i\omega t}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = -i^2\omega^2 \psi_0 e^{-i\omega t}$$

$$= -\omega^2 \psi_0 e^{-i\omega t} \quad (\because -i^2 = -1)$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi \quad \text{---} (3)$$

On putting the value of eqn (3) in eqn (1), we get,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{\omega^2}{v^2}\psi$$

$$\because \omega = 2\pi\upsilon$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{4\pi^2\upsilon^2}{v^2}\psi \quad \text{---} (4)$$

Again, $\because \lambda = v/\upsilon$

Therefore by eqn (4), we have,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2}\psi \quad \text{---}$$

But the De Broglie wave length associated with a particle, moving in the form of a wave is given by,

$$\lambda = \frac{h}{mv}$$

Therefore, by equation (5), we have,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{-4\pi^2}{\left(\frac{h}{mv}\right)^2}\psi$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{-4\pi^2 m^2 v^2}{h^2}\psi \quad (6)$$

If E is the total energy of the particle, then it is the sum of both the P.E. and K.E., that is,

$$E = V + \frac{1}{2}mv^2$$

$$\Rightarrow 2(E-V) = mv^2$$

$$\Rightarrow m^2v^2 = 2m(E-V)$$ [on multiply both sides by m]

On putting the value of m^2v^2 in eqn (6), we get,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{-4\pi^2 \cdot 2m(E-V)}{h^2}\psi$$

$$\Rightarrow \nabla^2\psi = -\frac{8\pi^2 m(E-V)}{h^2}\psi$$

where, $\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$, calle the Laplacian operator.

$$\therefore \nabla^2\psi + \frac{8\pi^2 m(E-V)}{h^2}\psi = 0$$

This is required Schrödinger's time indepe equation in three dimensions.

Q Discuss Schrodinger's Time dependent Wave function.

Ans:- Schrodinger's time dependent equation can be derived by eliminating E from the Schrodinger's time independent wave equation.

We have :-

$\psi(x,y,z,t) = \psi_0(x,y,z) \cdot e^{-i\omega t}$ ——— ①

On differentiating equation ① w.r.t. time we get,

$\frac{\partial \psi}{\partial t} = -i\omega \psi_0 e^{-i\omega t}$

$\Rightarrow \partial \psi/\partial t = -i 2\pi \nu \psi_0 e^{-i\omega t}$ $(\because \omega = 2\pi\nu)$

$\Rightarrow \partial \psi/\partial t = -i 2\pi (E/h) \psi_0 e^{-i\omega t}$ $(\because E = h\nu)$

$\Rightarrow \frac{\partial \psi}{\partial t} = -i\left(\frac{E}{h/2\pi}\right) \psi_0 e^{-i\omega t}$

$\Rightarrow \partial \psi/\partial t = -i\left(\frac{E}{\hbar}\right)\psi$ $\left[\because \psi = \psi_0 e^{-i\omega t} \ \& \ h/2\pi = \hbar\right]$

$\hbar$ is called reduced Plank constant

$\Rightarrow -\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = E\psi$

$\Rightarrow \frac{-i\hbar}{i \times i}\frac{\partial \psi}{\partial t} = E\psi$ (On multiplying and dividing the left part by i)

$\Rightarrow i\hbar\, \partial \psi/\partial t = E\psi$ ② $(\because i^2 = -1)$

We have from Schrodinger's time independent function -

$\nabla^2 \psi + \frac{2m(E-V)}{\hbar^2}\psi = 0$

$\Rightarrow \nabla^2 \psi = -2m/\hbar^2 \,(E\psi - V\psi)$

On replacing the value of $E\psi$ from eqn ② in the above equation we get,

$\nabla^2 \psi = -\frac{2m}{\hbar}\left(i\hbar \frac{\partial \psi}{\partial t} - V\psi\right)$

③

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t} - V\psi$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}} \quad — (3)$$

This is well known Schrodinger's time dependent wave equation. The above equation can also be written as -

$$H\psi = E\psi \quad — (4)$$

where $H = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)$, known as Hamiltonian operator and

$E = i\hbar\frac{\partial}{\partial t}$ is the energy operator.

Q Give some physical significance of a wave function ψ?

An. Physical significance of ψ are :-

i) In the matter waves, the quantities that varies periodically is called the wave function ψ.

ii) The wave function ψ is a complex quantity that means it includes both real and imaginary parts.

iii) Even though the wave function ψ by itself has no physical meaning the probability density $|\psi|^2 = \psi^*\psi$ is a real and positive quantity and is measurable.

iv) Probility density $|\psi|^2$ is the probability of finding the particle in a unit volume and it can have values anywhere between 0 and 1.

v) Higher values of probability density mean the probability of finding the particle within a given volume is more.

Q The wave function of a certain particle is

$$\psi = A\cos^2 x \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Determine:- i) The value of A.

ii) Probability that the particle can be found between $x=0$ and $x=\frac{\pi}{4}$.

Sol:- The given wave function is,

$$\psi = A\cos^2 x, \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

i) The value of A can be calculated by using

$$\int_{-\pi/2}^{\pi/2} |\psi|^2 dx = 1$$

$$\Rightarrow 2A^2 \int_0^{\pi/2} \cos^4 x\, dx = 1$$

$$\Rightarrow 2A^2 \times \frac{3\pi}{16} = 1 \Rightarrow A^2 = \frac{8}{3\pi}$$

$$\therefore A = \sqrt{8/3\pi}$$

ii) Calculation of probability:-

$$P = \int_0^{\pi/4} |\psi|^2 dx$$

$$= A^2 \int_0^{\pi/4} \cos^4 x\, dx$$

$$\Rightarrow P = \frac{8}{3\pi} \int_0^{\pi/4} \cos^4 x\, dx \quad \left(\because A^2 = \frac{8}{3\pi}\right)$$

$$\Rightarrow \boxed{P = 0.462}$$